A

COMMENTARY

ON

NEWTON'S PRINCIPIA.

WITH

A SUPPLEMENTARY VOLUME.

DESIGNED FOR THE USE OF STUDENTS AT THE UNIVERSITIES.

BY

J. M. F. WRIGHT, A. B.

LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE, AUTHOR OF SOLUTIONS OF THE CAMBRIDGE PROBLEMS, &c. &c.

IN TWO VOLUMES.

VOL. I.

LONDON:
PRINTED FOR T. T. & J. TEGG, 73, CHEAPSIDE;
AND RICHARD GRIFFIN & CO., GLASGOW.

MDCCCXXXIII.

GLASGOW:
GEORGE BROOKMAN, PRINTER, VILLAFIELD.

PREFACE.

The flattering manner in which the *Glasgow Edition* of Newton's Principia has been received, a second impression being already on the verge of publication, has induced the projectors and editor of that work, to render, as they humbly conceive, their labours still more acceptable, by presenting these additional volumes to the public. From amongst the several testimonies of the esteem in which their former endeavours have been held, it may suffice, to avoid the charge of self-eulogy, to select the following, which, coming from the high authority of *French* mathematical criticism, must be considered at once as the more decisive and impartial. It has been said by one of the first geometers of France, that "L'edition de Glasgow fait honneur aux presses de cette ville industrieuse. On peut affirmer que jamais l'art typographique ne rendit un plus bel hommage à la memoire de Newton. Le mérite de l'impression, quoique tres-remarquable, n'est pas ce que les editeurs ont recherché avec le plus de soin, pour tant le materiel de leur travail, ils pouvaient s'en rapporter à l'habilité de leur artistes: mais le choix des meilleures éditions, la revision la plus scrupuleuse du texte et des épreuves, la recherche attentive des fautes qui pourraient échapper même au lecteur studieux, et passer inaperçues ce travail consciencieux de l'intelligence et du savoir, voilà ce qui élève cette edition au-dessus de toutes celles qui l'ont précédée.

"Les editeurs de Glasgow ne s'etaient chargés que d'un travail de revision. *S'ils avaient conçu le projet d'améliorer et compléter l'œuvre des*

commentateurs, ils auraient sans doute employé, comme eux, les travaux des successeurs de Newton sur les questions traitées dans le livre des Principes.

"Les descendans de Newton sont nombreux, et leur genealogie est prouvée par des titres incontestibles; ceux qui vivent aujourd'hui verraient sans doute avec satisfaction que l'on formât un tableau de leur famille, en reunissant les productions les plus remarquables dont l'ouvrage de Newton a fourni le germe: que ce livre immortel soit entouré de tout ce l'on peut regarder comme ses developpemens: voilà son meilleur commentaire. *L'edition de Glasgow pourrait donc être continuée, et prodigieusement enrichie.*"

The same philosopher takes occasion again to remark, that "Le plus beau monument que l'on puisse élever à la gloire de Newton, c'est une bonne edition de ses ouvrages: et il est étonnant que les Anglais en aient laissé ce soin aux nations etrangères. Les presses de Glasgow viennent de reparer, en partie, le tort de la nation Anglaise: la nouvelle edition des *Principes* est effectivement *la plus belle, la plus correcte et la plus commode qui ait paru jusqu'ici.* La collation des anciennes editions, la revision des calculs, &c. ont été confiees á un habile mathematicien et rien n'a été négligé pour éviter toutes les erreurs et toutes les omissions.

"Il faut esperer que *les editeurs continueront leur belle entreprise, et qu'ils y seront assez encouragés pour nous donner, non seulement tous les ouvrages de Newton, mais ceux des savans qui ont complété ses travaux.*"

The encouragement here anticipated has not been withheld, nor has the idea of improving and completing the comments of "The Jesuits", contained in the Glasgow Newton, escaped us, inasmuch as long before these hints were promulgated, had the following work, which is composed principally as a succedaneum to the former, been planned, and partly written. It is at least, however, a pleasing confirmation of the justness of our own conceptions, to have encountered even at any time with these after-suggestions. The plan of the work is, nevertheless, in several respects, a deviation from that here so forcibly recommended.

The object of the first volume is, to make the text of the Principia, by

supplying numerous steps in the very concise demonstrations of the propositions, and illustrating them by every conceivable device, as easy as can be desired by students even of but moderate capacities. It is universally known, that Newton composed this wonderful work in a very hasty manner, merely selecting from a huge mass of papers such discoveries as would succeed each other as the connecting links of one vast chain, but without giving himself the trouble of explaining to the world the mode of fabricating those links. His comprehensive mind could, by the feeblest exertion of its powers, condense into one view many syllogisms of a proposition even heretofore uncontemplated. What difficulties, then, *to him* would seem his own discoveries? Surely none; and the modesty for which he is proverbially remarkable, gave him in his own estimation so little the advantage of the rest of created beings, that he deemed these difficulties as easy to others as to himself: the lamentable consequence of which humility has been, that he himself is scarcely comprehended at this day—a century from the birth of the Principia.

We have had, in the first place, the Lectures of Whiston, who descants not even respectably in his lectures delivered at Cambridge, upon the discoveries of his master. Then there follow even lower and less competent interpreters of this great prophet of science—for such Newton must have been held in those dark days of knowledge—whom it would be time mis-spent to dwell upon. But the first, it would seem, who properly estimated the Principia, was Clairaut. After a lapse of nearly half a century, this distinguished geometer not only acknowledged the truths of the Principia, but even extended the domain of Newton and of Mathematical Science. But even Clairaut did not condescend to explain his views and perceptions to the rest of mankind, farther than by publishing his own discoveries. For these we owe a vast debt of gratitude, but should have been still more highly benefited, had he bestowed upon us a sort of running Commentary on the Principia. It is generally supposed, indeed, that the greater portion of the Commentary called Madame Chastellet's, was due to Clairaut. The best things, however, of that work are alto-

gether unworthy of so great a master; at the most, showing the performance was not one of his own seeking. At any rate, this work does not deserve the name of a Commentary on the Principia. The same may safely be affirmed of many other productions intended to facilitate Newton. Pemberton's View, although a bulky tome, is little more than a eulogy. Maclaurin's speculations also do but little, elucidate the dark passages of the Principia, although written more immediately for that purpose. This is also a heavy unreadable performance, and not worthy a place on the same shelf with the other works of that great geometer. Another great mathematician, scarcely inferior to Maclaurin, has also laboured unprofitably in the same field. Emerson's Comments is a book as small in value as it is in bulk, affording no helps worth the perusal to the student. Thorpe's notes to the First Book of the Principia, however, are of a higher character, and in many instances do really facilitate the reading of Newton. Jebb's notes upon certain sections deserve the same commendation; and praise ought not to be withheld from several other commentators, who have more or less succeeded in making small portions of the Principia more accessible to the student—such as the Rev. Mr. Newton's work, Mr. Carr's, Mr. Wilkinson's, Mr. Lardner's, &c. It must be confessed, however, that all these fall far short in value of the very learned labours, contained in the Glasgow Newton, of the Jesuits Le Seur and Jacquier, and their great coadjutor. Much remained, however, to be added even to this erudite production, and subsequently to its first appearance much has been excogitated, principally by the mathematicians of Cambridge, that focus of science, and native land of the Principia, of which, in the composition of the following pages, the author has liberally availed himself. The most valuable matter thus afforded are the Tutorial MSS. in circulation at Cambridge. Of these, which are used in explaining Newton to the students by the Private Tutors there, the author confesses to have had abundance, and also to have used them so far as seemed auxiliary to his own resources. But at the same time it must be remarked, that little has been the assistance hence derived, or, indeed, from all

other known sources, which from the first have been constantly at command.

The plan of the work being to make those parts of Newton easy which are required to be read at Cambridge and Dublin, that portion of the Principia which is better read in the elementary works on Mechanics, viz. the preliminary Definitions, Laws of Motion, and their Corollaries, has been disregarded. For like reasons the fourth and fifth sections have been but little dwelt upon. The eleventh section and third book have not met with the attention their importance and intricacy would seem to demand, partly from the circumstance of an excellent Treatise on Physics, by Mr. Airey, having superseded the necessity of such labours; and partly because in the second volume the reader will find the same subjects treated after the easier and more comprehensive methods of Laplace.

The first section of the first book has been explained at great length, and it is presumed that, for the first time, the true principles of what has been so long a subject of contention in the scientific world, have there been fully established. It is humbly thought (for in these intricate speculations it is folly to be proudly confident), that what has been considered in so many lights and so variously denominated Fluxions, Ultimate Ratios, Differential Calculus, Calculus of Derivations, &c. &c. is here laid down on a basis too firm to be shaken by future controversy. It is also hoped that the text of this section, hitherto held almost impenetrably obscure, is now laid open to the view of most students. The same merit it is with some confidence anticipated will be awarded to the illustrations of the 2nd, 3rd, 6th, 7th, 8th, and 9th sections, which, although not so recondite, require much explanation, and many of the steps to be supplied in the demonstration of almost every proposition. Many of the things in the first volume are new to the author, but very probably not original in reality—so vast and various are the results of science already accumulated. Suffice it to observe, that if they prove useful in unlocking the treasures of the Principia, the author will rest satisfied with the meed of approbation, which he will to that extent have earned from a discriminating and impartial public.

The second volume is designed to form a sort of Appendix or Supplement to the Principia. It gives the principal discoveries of Laplace, and, indeed, will be found of great service, as an introduction to the entire perusal of the immortal work of that author—the Mecanique Celeste. This volume is prefaced by much useful matter relative to the Integration of Partial Differences and other difficult branches of Abstract Mathematics, those powerful auxiliaries in the higher departments of Physical Astronomy, and which appear in almost every page of the Mecanique Celeste. These and other preparations, designed to facilitate the comprehension of the Newton of these days, will, it is presumed, be found fully acceptable to the more advanced readers, who may be prosecuting researches even in the remotest and most hidden receptacles of science; and, indeed, the author trusts he is by no means unreasonably exorbitant in his expectations, when he predicates of himself that throughout the undertaking he has proved himself a labourer not unworthy of reward.

THE AUTHOR.

TO THE TUTORS

OF THE SEVERAL COLLEGES AT CAMBRIDGE,

THESE PAGES,

WHICH WERE COMPOSED WITH THE VIEW

OF PROMOTING THE STUDIES

OVER WHICH THEY SO ABLY PRESIDE,

ARE RESPECTFULLY INSCRIBED

BY THEIR DEVOTED SERVANT,

THE AUTHOR.

A COMMENTARY

ON

NEWTON'S PRINCIPIA.

SECTION I. BOOK I.

1. This section is introductory to the succeeding part of the work. It comprehends the substance of the method of Exhaustions of the Ancients, and also of the Modern Theories, variously denominated *Fluxions*, *Differential Calculus*, *Calculus of Derivations*, *Functions*, &c. &c. Like them it treats of the relations which Indefinite quantities bear to one another, and conducts in general by a nearer route to precisely the same results.

2. In what precedes this section, *finite* quantities only are considered, such as the spaces described by bodies moving uniformly in *finite* times with *finite* velocities; or at most, those described by bodies whose motions are *uniformly accelerated*. But what follows relates to the motions of bodies accelerated according to various hypotheses, and requires the consideration of quantities indefinitely small or great, or of such whose Ratios, by their decrease or increase, continually approximate to certain Limiting Values, but which they cannot reach be the quantities ever so much diminished or augmented. These *Limiting Ratios* are called by Newton, " Prime and Ultimate Ratios," *Prime Ratio* meaning the Limit from which the Ratio of two quantities *diverges*, and *Ultimate Ratio* that towards which the Ratio *converges*. To prevent ambiguity, the term *Limiting Ratio* will subsequently be used throughout this Commentary.

LEMMA I.

3. QUANTITIES AND THE RATIOS OF QUANTITIES.] Hereby Newton would infer the truth of the Lemma not only for quantities mensurable by *Integers*, but also for such as may be denoted by *Vulgar Fractions*. The necessity or use of the distinction is none; there being just as much reason for specifying all other sorts of quantities. The truth of the LEMMA does not depend upon the *species* of quantities, but upon their conformity with the following conditions, viz.

4. That they *tend continually to equality, and approach nearer to each other than by any given difference*. They must tend *continually* to equality, that is, every Ratio of their successive corresponding values must be nearer and nearer a Ratio of Equality, the number of these convergencies being without end. By *given difference* is merely meant any that can be assigned or proposed.

5. FINITE TIME.] Newton obviously introduces the idea of time in this enunciation, to show illustratively that he supposes the quantities to converge continually to equality, without ever actually *reaching or passing* that state; and since to fix such an idea, he says, "before the end of that time," it was moreover necessary to consider the time *Finite*. Hence our author would avoid the charge of "*Fallacia Suppositionis*," or of "*shifting the hypothesis*." For it is contended that if you frame certain relations between actual quantities, and afterwards deduce conclusions from such relations on the supposition of the quantities having vanished, such conclusions are illogically deduced, and ought no more to subsist than the quantities themselves.

In the Scholium at the end of this Section he is more explicit. He says, *The ultimate Ratios, in which quantities vanish, are not in reality the Ratios of Ultimate quantities; but the Limits to which the Ratios of quantities continually decreasing always approach; which they never can pass beyond or arrive at, unless the quantities are continually and indefinitely diminished.* After all, however, neither our Author himself nor any of his Commentators, though much has been advanced upon the subject, has obviated this objection. Bishop Berkeley's ingenious criticisms in the Analyst remain to this day unanswered. He therein facetiously denominates the results, obtained from the supposition that the quantities, before

considered finite and real, have vanished, the "*Ghosts of Departed Quantities*;" and it must be admitted there is reason as well as wit in the appellation. The fact is, Newton himself, if we may judge from his own words in the above cited Scholium, where he says, "If two quantities, whose DIFFERENCE IS GIVEN are augmented continually, their Ultimate Ratio will be a Ratio of Equality," had no knowledge of the *true nature* of his Method of Prime and Ultimate Ratios. If there be meaning in words, he plainly supposes in this passage, a mere Approximation to be the same with an Ultimate Ratio. He loses sight of the condition expressed in Lemma I. namely, *that the quantities tend to equality nearer than by any assignable difference*, by supposing the difference of the quantities continually augmented to be *given*, or always the same. In this sense the whole Earth, compared with the whole Earth minus a grain of sand, would constitute an Ultimate Ratio of equality; whereas so long as any, the minutest difference exists between two quantities, they cannot be said to be more than *nearly* equal. But it is now to be shown, that

6. *If two quantities tend continually to equality, and approach to one another nearer than by any assignable difference, their Ratio is* ULTIMATELY *a Ratio of* ABSOLUTE *equality.* This may be demonstrated as follows, even without supposing the quantities ultimately evanescent.

It is acknowledged by all writers on Algebra, and indeed self-evident, that if in any equation put $= 0$, there be quantities absolutely different in kind, the aggregate of each species is separately equal to 0. For example, if

$$A + a + B\sqrt{2} + b\sqrt{2} + C\sqrt{-1} = 0,$$

since $A + a$ is *rational*, $(B + b)\sqrt{2}$ *surd* and $C\sqrt{-1}$ *imaginary*, they cannot in any way destroy one another by the opposition of signs, and therefore

$$A + a = 0,\ B + b = 0,\ C = 0.$$

In the same manner, if *logarithms*, *exponentials*, or any other quantities differing essentially from one another constitute an equation like the above, they must separately be equal to 0. This being premised, let L, L' denote the Limits, whatever they are, towards which the quantities $L + l$, $L' + l'$ continually converge, and suppose their difference, in any state of the convergence, to be D. Then

$$L + l - L' - l' = D,$$

$$\text{or } L - L' + l - l' - D = 0,$$

and since L, L' are fixed and definite, and l, l', D always variable, the former are independent of the latter, and we have

$$L - L' = 0, \text{ or } \frac{L}{L'} = 1, \text{ accurately. } \quad Q. e. d.$$

This way of considering the question, it is presumed, will be deemed free from every objection. The principle upon which it rests depending upon the *nature* of the variable quantities, and not upon their evanescence, (as it is equally true even for constant quantities provided they be of different natures), it is hoped we have at length hit upon the true and logical method of expounding the doctrine of *Prime and Ultimate Ratios*, or of *Fluxions*, or of the *Differential Calculus*, &c.

It may be here remarked, in passing, that the Method of *Indeterminate Coefficients*, which is at bottom the same as that of Prime and Ultimate Ratios, is treated illogically in most books of Algebra. Instead of "shifting the hypothesis," as is done in Wood, Bonnycastle and others, by making $x = 0$, in the equation

$$a + b\,x + c\,x^2 + d\,x^3 + \ldots\ldots = 0,$$

it is sufficient to know that each term x being indefinitely *variable*, is heterogeneous compared with the rest, and consequently that each term must equal 0.

7. Having established the truth of LEMMA I. on incontestable principles, we proceed to make such applications as may produce results useful to our subsequent comments. As these applications relate to the Limits of the Ratios of the *Differences* of Quantities, we shall term, after Leibnitz, the Method of Prime and Ultimate Ratios,

THE DIFFERENTIAL CALCULUS.

8. According to the established notation, let a, b, c, &c., denote constant quantities, and z, y, x, &c., variable ones. Also let Δz, Δy, Δx, &c., represent the difference between any two values of z, y, x, &c., respectively.

9. *Required the Limiting or Ultimate Ratio of* $\Delta (a\,x)$ and Δx, i. e. *the Limit of the Difference of a Rectangle having one side* (a) *constant, and the other* (x) *variable, and of the Difference of the variable side.*

Let L be the Limit sought, and L + l any value whatever of the varying Ratio. Then

$$L + = l\frac{\Delta\,(a\,x)}{\Delta x} = \frac{a\,(x + \Delta x) - a\,x}{\Delta x} = a. \quad \therefore \text{ by No. 7}$$

$$L = a.$$

In this instance the Ratio is the same for all values of x. But if in the Limit we change the characteristic Δ into d, we have

$$\left.\begin{array}{l} \frac{d(a x)}{d x} = a \\ \text{or} \\ d(a x) = a\, d x \end{array}\right\} \quad \ldots\ldots \quad (b)$$

d (a x), d x being called the *Differentials* of a x and x respectively.

10. *Required the Limit of* $\frac{\Delta(x^2)}{\Delta x}$.

Let L be the Limit required, and L + l the value of the Ratio generally. Then

$$L + l = \frac{\Delta(x^2)}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} =$$

$$\frac{2 x \Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x.$$

$$\therefore L - 2x + l - \Delta x = 0$$

and since L — 2 x and l — Δ x are heterogeneous

$$L - 2x = 0,$$

or

$$L = x 2$$

and $\therefore$

$$\frac{d(x^2)}{d x} = 2x$$

or

$$d(x^2) = 2 x d x \quad \ldots\ldots\ldots\ldots \quad (c)$$

11. *Generally, required the Limit of* $\frac{\Delta(x^n)}{\Delta x}$.

Let L and L + l be the Limit of the Ratio and the Ratio itself respectively. Then

$$L + l = \frac{\Delta(x^n)}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= n x^{n-1} + \frac{n.(n-1)}{2}.x^{n-2}\Delta x + \&c.$$

and L — n x^{n-1} being essentially different from the other terms of the series and from l, we have

$$\frac{d(x^n)}{d x} = L = n x^{n-1} \text{ or } d(x^n) = n x^{n-1} d x \quad \ldots\ldots \quad (d)$$

or in words,

The Differential of any power or root of a variable quantity is equal to the product of the Differential of the quantity itself, the same power or root MINUS *one of the quantity, and the index of the power or root.*

We have here supposed the Binomial Theorem as fully established by Algebra. It may, however, easily be demonstrated by the general principle explained in (7).

12. From 9 and 11 we get

$$d\,(a\,x^n) = n\,a\,x^{n-1}\,d\,x \quad . \; . \; . \; . \; . \; . \; . \quad (e)$$

13. *Required the Limit of* $\frac{\Delta\,(a + b\,x^n + c\,x^m + e\,x^p + \&c.)}{\Delta\,x}$

Let L be the Limit sought, and L + l the variable Ratio of the finite differences; then

$$L + l = \frac{\Delta\,(a + b\,x^n + c\,x^m + \&c.)}{\Delta\,x}$$

$$= \frac{a + b\,(x + \Delta\,x)^n + c\,(x + \Delta\,x)^m + \&c. - a - b\,x^n - c\,x^m - \&c.}{\Delta\,x}$$

$$= n\,b\,x^{n-1} + m\,c\,x^{m-1} + \&c. + P\,\Delta\,x + Q\,(\Delta\,x)^2 + \&c.$$

P, Q, &c. being the coefficients of $\Delta\,x$, $\Delta\,x^2$ + &c. And equating the homogeneous determinate quantities, we have

$$\frac{d(a + b x^n + c x^m + \&c.)}{d\,x} = L = n b x^{n-1} + m c x^{m-1} + p e x^{p-1} + \&c. \ldots (f)$$

14. *Required the Limit of* $\frac{\Delta\,(a + b\,x^n + c\,x^m + \&c.)^r}{\Delta\,x}$.

By 11 we have

$$\frac{d.\,(a + b\,x^n + c\,x^m + \&c.)^r}{d\,(a + b\,x^n + \&c.)} = r\,(a + b\,x^n + c\,x^m + \&c.)^{r-1}$$

and by 13

$$d\,(a + b\,x^n + c\,x^m + \&c.) = (n\,b\,x^{n-1} + m\,c\,x^{m-1} + \&c.)\,d\,x$$

$$\therefore \frac{d(a + bx^n + cx^m + \&c.)^r}{d\,x} = r(nbx^{n-1} + mcx^{m-1} + \&c.)(a + bx^n + \&c.)^{r-1} \ldots (g)$$

the Limiting Ratio of the Finite Differences $\Delta\,(a + b\,x^n + c\,x^m + \&c.)$, $\Delta\,x$, that is the Ratio of the Differentials of $a + b\,x^n + c\,x^m + \&c.$, and x.

15. *Required the Ratio of the Differentials of* $\frac{A + B\,x^n + C\,x^m + \&c.}{a + b\,x^\nu + C\,x^\mu + \&c.}$ and x, *or the Limiting Ratio of their Finite Differences.*

Let L be the Limit required, and L + l the varying Ratio. Then

$$L + l = \frac{A + B\,(x + \Delta\,x)^n + C\,(x + \Delta\,x)^m + \&c.}{a + b\,(x + \Delta\,x)^\nu + c\,(x + \Delta\,x)^\mu + \&c.} - \frac{A + B\,x^n + \&c.}{a + b\,x^\nu + \&c.}$$

which being expanded by the Binomial Theorem, and properly reduced gives

$$L \times (a + b x^{\nu} + \&c.)^2 + L \times \{P. \Delta x + Q (\Delta x)^2 + \&c. + l \times \{a + b x^{\nu} + \&c. + P. \Delta x + Q (\Delta x)^2 + \&c.\} = (a + b x^{\nu} + c x^{\mu} + \&c.) \times (n B x^{n-1} + m C x^{m-1} + \&c.) - (A + B x^n + C x^m + \&c.) \times (\nu b x^{\nu - 1} + \mu c x^{\mu - 1} + \&c.) + P'. \Delta x + Q' (\Delta x)^2 + \&c.$$

P, Q, P′, Q′ &c. being coefficients of Δx, $(\Delta x)^2$ &c. and independent of them.

Now equating those *homogeneous* terms which are independent of the powers of Δx, we get

$$L(a + b x^{\nu} + \&c.)^2 = (a + b x^{\nu} + \&c.) - (n B x^{n-1} + m C x^{m-1} + \&c.) - (A + B x^n + C x^m + \&c.) - (\nu b x^{\nu - 1} + \mu c x^{\mu - 1} + \&c.)$$

and putting $u = \frac{A + B x^n + C x^m + \&c.}{a + b x^{\nu} + c x^{\mu} + \&c.}$ we have finally

$\frac{d u}{d x} = L$, and therefore $\frac{d u}{d x} =$

$$\frac{(a + b x^{\nu} + \&c.)(n B x +^{n-1} m C x^{m-1} + \&c.) - (A + B x^n + \&c.)(\nu b x^{\nu-1} + \mu c x^{\mu - 1} + \&c.)}{(a + b x^{\nu} + c x^{\mu} + \&c.)^2}$$

the Ratio required.

16. Hence and from 11 we have the Ratio of the Differentials of $\frac{(A + B x^n + C x^m + \&c.)^p}{(a + b x^{\nu} + c x^{\mu} + \&c.)^q}$ and x; and in short, from what has already been delivered it is easy to obtain the Ratio of the Differentials of any *Algebraic Function whatever of one variable* and of that variable.

N. B. By *Function of a variable* is meant a quantity anyhow involving that variable. The term was first used to denote the Powers of a quantity, as x^2, x^3, &c. But it is now used in the general sense.

The quantities next to Algebraical ones, in point of simplicity, are Exponential Functions; and we therefore proceed to the investigation of their Differentials.

17. *Required the Ratio of the Differentials of* a^x and x; *or the Limiting Ratio of their Differences.*

Let L be the required Limit and L + l the varying Ratio; then

$$L + l = \frac{\Delta (a^x)}{\Delta x} = \frac{a^{x + \Delta x} - a^x}{\Delta x}$$
$$= a^x \times \frac{a^{\Delta x} - 1}{\Delta x}.$$

But since

$$a^y = (1 + a - 1)^y$$

$$= 1 + y(a-1) + \frac{y.(y-1)}{2}.(a-1)^2 +$$

$$\frac{y.(y-1)(y-2)}{2.3}(a-1)^3 + \&c.,$$

it is easily seen that the coefficient of y in the expansion is

$$a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.$$

Hence

$$L + 1 = \frac{a^x}{\Delta x}\{(a-1-\frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.)\Delta x + P(\Delta x)^2 + \&c.\}$$

and equating homogeneous quantities, we have

$$\frac{d.(a^x)}{dx} = L = \{a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.\}\ a^x$$

$$= A\,a^x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

or the Ratio of the Differentials of any Exponential and its exponent is equal to the product of the Exponential and a constant Quantity.

Hence and from the preceding articles, *the Ratio of the Differentials of any Algebraic Function of Exponentials having the same variable index, may be found.* The Student may find abundance of practice in the Collection of Examples of the Differential and Integral Calculus, by Messrs. Peacock, Herschel and Babbage.

Before we proceed farther in Differentiation of quantities, let us investigate the nature of the constant A which enters the equation (h).

For that purpose, let (the two first terms have been already found)

$$a^x = 1 + Ax + Px^2 + Qx^3 + \&c.$$

Then, by 13,

$$\frac{d(a^x)}{dx} = A + 2Px + 3Qx^2 + 4Rx^3 + \&c.$$

But by equation (h)

$$\frac{d(a^x)}{dx} \text{ also} = A\,a^x$$

$$= A + A^2x + APx^2 + AQx^3 + \&c.$$

$$\therefore A + 2Px + 3Qx^2 + 4Rx^3 + \&c. = A + A^2x + APx^2 + \&c.$$

and equating *homogeneous quantities,* we get

$$2P = A^2,\ 3Q = AP,\ 4R = AQ,\ \&c. = \&c.$$

whence

$$P = \frac{A^2}{2}, Q = \frac{A P}{3} = \frac{A^3}{2.3}, R = \frac{A Q}{4} = \frac{A^4}{2.3.4} \text{ \&c. \&c.}$$

Therefore,

$$a^x = 1 + A x + \frac{A^2}{2} x^2 + \frac{A^3}{2.3} x^3 + \frac{A^4}{2.3.4} x^4 + \text{\&c.}$$

Again, put $A x = 1$, then

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{\&c.}$$

$$= 2.718281828459 \text{ as is easily calculated}$$

$$= e$$

by supposition. Hence

$$A = \frac{\log. a}{\log. e} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (k)$$

$$\therefore a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.} = \frac{\log. a}{\log. e} = l.\, a$$

for the system whose base is e, l being the characteristic of that system.

This system being that which gives

$$e - 1 - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} - \text{\&c.} = 1$$

is called Natural from being the most simple.

Hence the equation (h) becomes

$$\frac{d\,(a^x)}{d\,x} = l\,a \times a^x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (l)$$

17 a. *Required the Ratio of the Differentials of* l (x) and x.

Let $l\,x = u$. Then $e^u = x$

$$\therefore d\,x = d\,(e^u) = l\,e \times e^u\,d\,u = e^u\,d\,u, \text{ by 16}$$

$$\therefore \frac{d\,(l\,x)}{d\,x} = \frac{1}{e^u} = \frac{1}{x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (m)$$

In any other system whose base is a, we have $\log.\,(x) = \frac{l\,x}{l\,a}$.

$$\therefore \frac{d \log. x}{d\,x} = \frac{1}{l\,a} \times \frac{1}{x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (n)$$

We are now prepared to differentiate any Algebraic, or Exponential Functions of Logarithmic Functions, provided there be involved but one variable.

Before we differentiate circular functions, viz. the sines, cosines, tangents, &c., of circular arcs, we shall proceed with our comments on the text as far as Lemma VIII.

LEMMA II.

18. In No. 6, calling L and L′ Limits of the circumscribed and inscribed rectilinear figures, and L + l, L′ + l′ any other values of them, whose variable difference is D, the *absolute* equality of L and L′ is clearly demonstrated, without the supposition of the bases A B, B C, C D, D E, being infinitely diminished in number and augmented in magnitude. In the view there taken of the subject, it is necessary merely to suppose them *variable.*

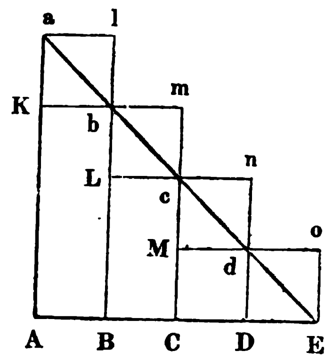
LEMMA III.

19. This LEMMA is also demonstrable by the same process in No. 6, as LEMMA II.

Cor. 1. The rectilinear figures cannot possibly *coincide* with the curvilinear figure, because the rectilinear boundaries a l b m c n d o E, a K b L c M d D E cut the curve a b E in the points a, b, c, d, E in finite angles. The learned Jesuits, Jacquier and Le Seur, in endeavouring to remove this difficulty, suppose the four points a, l, b, K to coincide, and thus to form a small element of the curve. But this is the language of Indivisibles, and quite inadmissible. It is plain that no *straight line*, or combination of straight lines, can form a *curve line*, so long as we understand by a *straight line* "that which lies evenly between its extreme points," and by a *curve line*, "that which does not lie evenly between its extreme points;" for otherwise it would be possible for a line to be straight and not straight at the same time. The truth is manifestly this. The Limiting Ratio of the inscribed and circumscribed figures is that of equality, because they continually tend to a fixed *area*, viz. that of the given intermediate curve. But although this intermediate curvilinear area, is the Limit towards which the rectilinear areas continually tend and approach nearer than by any difference; yet it does not follow that the rectilinear boundaries also tend to the curvilinear one as a limit. The rectilinear boundaries are, in fact, entirely *heterogeneous* with the intermediate one, and consequently cannot be equal to it, nor coincide therewith. We will now clear up the above, and at the same time introduce a striking illustration of the necessity there exists, of taking into consideration the *nature* of quantities, rather than their evanescence or infinitesimality.

Take the simplest example of Lemma II., in the case of the right-angled triangle a E A, having its two legs A a, A E equal.

The figure being constructed as in the text of Lemma II, it follows from that Lemma, that the Ultimate Ratio of the inscribed and circumscribed figures is a ratio of equality; and moreover it would also follow from *Cor.* 1. that either of these coincided ultimately with the triangle a E A. Hence then the exterior boundary a l b m c n d o E coincides exactly with a E *ultimately*, and they are consequently equal in the Limit. As we have only straight lines to deal with in this example, let us try to ascertain the exact ratio of a E to the exterior boundary.



If n be the indefinite number of equal bases A B, B C, &c., it is evident, since A a = A E, that the whole length of a l b m c n d o E = 2 n × A B. Also since a b = b c = &c. $= \sqrt{\mathrm{al}^2 + \mathrm{bl}^2} = \sqrt{2}$. A B, we have a E = n $\sqrt{2}$. A B.

Consequently,

a l b m c n d o E : a E : : 2 : $\sqrt{2}$: : $\sqrt{2}$: 1.

Hence it is plain the exterior boundary cannot possibly coincide with a E. Other examples might be adduced, but it must now be sufficiently clear, that Newton confounded the *ultimate equality* of the inscribed and circumscribed figures, to the intermediate one, with their actual coincidence, merely from deducing their Ratios on principles of approximation or rather of Exhaustion, instead of those, as explained in No. 6; which relate to the *homogeneity* of the quantities. In the above example the boundaries being *heterogeneous* inasmuch as they are *incommensurable*, cannot be compared as to magnitude, and unless lines are absolutely equal, it is not easy to believe in their coincidence.

Profound as our veneration is, and ought to be, for the Great Father of Mathematical Science, we must occasionally perhaps find fault with his obscurities. But it shall be done with great caution, and only with the view of removing them, in order to render accessible to students in general, the comprehension of "This greatest monument of human genius."

20. *Cor.* 2. 3. and 4. will be explained under Lemma VII, which relates to the Limits of the Ratios of the chord, tangent and the arc.

LEMMA IV.

21. Let the areas of the parallelograms inscribed in the two figures be denoted by

$$P, Q, R, \&c.$$
$$p, q, r, \&c.$$

respectively; and let them be such that

$$P : p :: Q : q :: R : r, \&c. :: m : n.$$

Then by compounding these equal ratios, we get

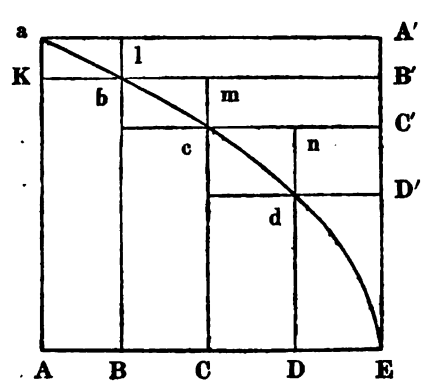
$$P + Q + R + \ldots\ldots : p + q + r + \ldots\ldots :: m : n$$

But $P + Q + R \ldots\ldots$ and $p + q + r + \ldots\ldots$ have with the curvilinear areas an ultimate ratio of equality. Consequently these curvilinear areas are in the given ratio of $m : n$.

Hence may be found the areas of certain curves, by comparing their incremental rectangles with those of a known area.

Ex. 1. *Required the area of the common Apollonian parabola comprised between its vertex and a given ordinate.*

Let a c E be the parabola, whose vertex is E, axis E A and *Latus-Rectum* $= a$. Then A A′ being its circumscribing rectangle, let any number of rectangles vertically opposite to one another be inscribed in the areas a E A, a E A′, viz. A b, b A′; B c, c B′, &c.



And since

$$A\,b = A\,K.\,A\,B$$

$$A'\,b = A'\,l.\,A'\,B' = \frac{A\,K^2}{a}.\,A'\,B'$$

from the equation to the parabola.

$$\therefore \frac{A\,b}{A'\,b} = \frac{a.\,A\,B}{A\,K.\,A'\,B'}$$

Also

$$(A\,a)^2 - B\,b^2 = a \times A\,E - a \times B\,E = a \times A\,B$$

or

$$(A\,a + B\,b) \times A'\,B' = a \times A\,B$$

$$\therefore \frac{a \times A B}{A' B'} = A a + B b$$

$$\therefore \frac{A b}{A' b} = \frac{A a + B b}{B b} = \frac{2 B b + K a}{B b} = 2 + \frac{K a}{B b}$$

Hence, since in the Limit $\frac{A b}{A' b}$ becomes fixed or of the same nature with the first term, we have

$$\frac{A b}{A' b} = 2$$

ultimately.

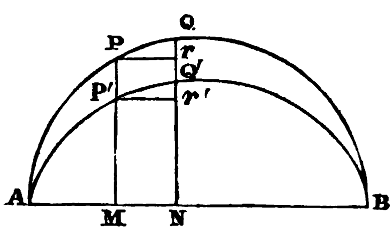
And the same may be shown of all other corresponding pairs of rectangles; consequently by Lemma IV.

a E A : a E A' : : 2 : 1

∴ a E A : rectangle A A' : : 2 : 3.

or the area of a parabola is equal to two thirds of its circumscribing rectangle.

Ex. 2. *To compare the area of a semiellipse with that of a semicircle described on the same diameter.*



Taking any two corresponding inscribed rectangles P N, P' N; we have

P N : P' N : : P M : P' M : : a : b

a and b being the semiaxes major and minor of the ellipse; and all other corresponding pairs of inscribed rectangles have the same constant ratio; consequently by Lemma IV, *the semicircle has to the semiellipse the ratio of the major to the minor axis.*

As another example, the student may compare the area of a cycloid with that of its circumscribing rectangle, in a manner very similar to Ex. 1.

This method of squaring curves is very limited in its application. In the progress of our remarks upon this section, we shall have to exhibit a general way of attaining that object.

LEMMA V.

22. For the definition of similar rectilinear figures, and the truth of this LEMMA as it applies to them, see Euclid's Elements B. VI, Prop. 4, 19 and 20.

The farther consideration of this LEMMA must be deferred to the explanation of LEMMA VII.

LEMMA VI.

23. In the demonstration of this LEMMA, "*Continued Curvature*" at any point, is tacitly defined to be *such, that the arc does not make with the tangent at that point, an angle equal to a finite rectilinear angle.*

In a Commentary on this LEMMA if the demonstration be admitted, any other definition than this is plainly inadmissible, and yet several of the Annotators have stretched their ingenuity to substitute notions of continued curvature, wholly inconsistent with the above. The fact is, this LEMMA is so exceedingly obscure, that it is difficult to make any thing of it. In the enunciation, Newton speaks of the *angle between the chord and tangent ultimately vanishing*, and in the demonstration, it is the angle between the *arc and tangent* that must vanish ultimately. So that in the Limit, it would seem, the *arc and chord* actually coincide. This has not yet been established. In LEMMA III, *Cor.* 2, the coincidence ultimately of a chord and its arc is implied; but this conclusion by no means follows from the LEMMA itself, as may easily be gathered from No. 19. The very thing to be proved by aid of this LEMMA is, that the Ultimate Ratio of the chord to the arc is a ratio of equality, it being merely subsidiary to LEMMA VII. But if it be already considered that they *coincide*, of course they are equal, and LEMMA VII becomes nothing less than "*argumentum in circulo.*"

Newton introduces the idea of curves of "*continued curvature,*" *or such as make no angle with the tangent,* to intimate that this LEMMA does not apply to curves of *non-continued curvature*, or *to such as do make a finite angle with the tangent.* At least this is the plain meaning of his words. But it may be asked, are there any curves whose tangents are inclined to them? The question can only be resolved, by again admitting

the arc to be ultimately coincident with the chord; and by then showing, that curves may be imagined whose chord and tangent ultimately shall be inclined at a finite angle. The Ellipse, for instance, whose minor axis is *indefinitely less* than its major axis, is a curve of that kind; for taking the tangent at the vertex, and putting a, b, for the semiaxes, and y, x, for the ordinate and abscissa, we have

$$y^2 = \frac{b^2}{a^2} \times (2\,a\,x - x^2)$$

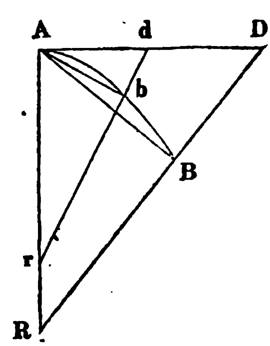
and

$$\frac{y}{x} = \frac{b}{a}\sqrt{\frac{2\,a}{x} \times 1} = \frac{b}{a\sqrt{x}}\sqrt{2\,a - x}$$

$\therefore$ since b is indefinitely smaller than a $\sqrt{x}$, x is indefinitely greater than y, and supposing y to be the tangent cut off by the secant x parallel to the axis, x and y are sides of a right angled $\triangle$, whose hypothenuse is the chord. Hence it is plain the $\angle$ opposite x is ultimately indefinitely greater than the $\angle$ opposite to y. But they are together equal to a right angle. Consequently the angle opposite x, or that between the chord and tangent, is ultimately finite. Other cases might be adduced, but enough has been said upon what it appears impossible to explain and establish as logical and direct demonstration. We confess our inability to do this, and feel pretty confident the critics will not accomplish it.

24. Having exposed the fallacy of Newton's reasoning in the proof of this Lemma, we shall now attempt something by way of substitute.

Let A D *be the tangent to the curve at the point* A, *and* A B *its chord. Then if* B *be supposed to move indefinitely near to* A, *the angle* B A D *shall indefinitely decrease, provided the curvature be not indefinitely great.*



Draw R D passing through B at right angles to A B, and meeting the tangent A D and normal A R in the points D and R respectively. Then since the angle B A D equals the angle A R B, if A R B decrease indefinitely when B approaches A; that is, if A R become indefinitely greater than A B; or which is the same thing, if the curvature at A, be not indefinitely great; the angle B A D also decreases indefinitely. Q. e. d.

We have already explained, by an example in the last article, what is

meant by curvature indefinitely great. It is the same with Newton's expression "continued curvature." The subject will be discussed at length under LEMMA XI.

As vanishing quantities are objectionable on account of their nothingness as it has already been hinted, and it being sufficient to consider variable quantities, to get their limiting ratios, *as capable of indefinite diminution*, the above enunciation has been somewhat modified to suit those views.

LEMMA VII.

25. This LEMMA, *supposing the two preceding ones to have been fully established*, would have been a masterpiece of ingenuity and elegance. By the aid of the proportionality of the homologous sides of similar curves, our author has exhibited quantities evanescent by others of any finite magnitude whatever, apparently a most ingenious device, and calculated to obviate all objections. But in the course of our remarks, it will be shown that LEMMA V cannot be demonstrated without the aid of this LEMMA.

First, by supposing A d, A b always finite, the angles at d and b and therefore those at D and B which are equal to the former are virtually considered finite, or R D cuts the chord and tangent at finite angles.

Hence the elaborate note upon this subject of Le Seur and Jacquier is rendered valueless as a direct comment.

Secondly. In the construction of the figure in this LEMMA, the description of a figure *similar* to any given one, is taken for granted. But the student would perhaps like to know how this can be effected.

LEMMA V, which is only enunciated, from being supposed to be a mere corollary to LEMMA III and LEMMA IV, would afford the means immediately, were it thence legitimately deduced. But we have clearly shown (Art. 19.) that rectilinear boundaries, consisting of lines cutting the intermediate curve ultimately *at finite angles*, cannot be equal ultimately to the curvilinear one, and thence we show that the boundaries formed by the chords or tangents, as stated in LEMMA III, *Cor.* 2 and 3, are not ultimately equal, by *consequence of that LEMMA*, to the curvilinear one.

Newton in *Cor.* 1, LEMMA III, asserts the ultimate coincidence, and therefore equality of the rectilinear boundary whose component lines cut the curve at finite angles, and thence would establish the succeeding cor-

ollaries *a fortiori*. But the truth is that the curvilinear boundary is the limit, as to magnitude, or length, of the tangential and chordal boundaries; although in the other case, it is a limit merely in respect of area. Yet, we repeat it, that LEMMA V cannot be made to follow from the LEMMAS preceding it. According to Newton's *implied* definition of similar curves, as explained in the note of Le Seur and Jacquier, *they are the curvilinear limits of similar rectilinear figures.* So they might be considered, if it were already demonstrated that the limiting ratio of the chord and arc is a ratio of equality; but this belongs to LEMMA VII. Newton himself and all the commentators whom we have perused, have thus committed a solecism. Even the best Cambridge MSS. and we have seen many belonging to the most celebrated private as well as college tutors in that learned university, have the same error. Nay most of them are still more inconsistent. They give definitions of similar curves wholly different from Newton's notion of them, and yet endeavour to prove LEMMA V, by aid of LEMMA VII. For the verification of these assertions, which may else appear presumptuously gratuitous, let the Cantabs peruse their MSS. The origin of all this may be traced to the falsely deduced *ultimate coincidence* of the curvilinear and rectilinear boundaries, in the corollaries of LEMMA III. See Art. 19.

We now give a demonstration of the LEMMA without the assistance of similar curves, and yet independently of quantities actually evanescent.

By hypothesis the secant R D cuts the chord and tangent at finite angles. Hence, since

$$A + B + D = 180^\circ$$

$$\therefore B + D = 180^\circ - A$$

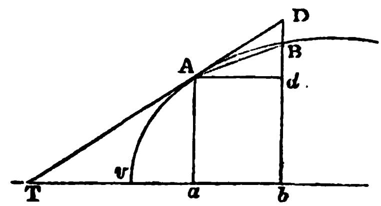
$$\text{or } L + l + L' + l' = 180^\circ - A$$

L and L′ being the limits of B and D and l, l′ their variable parts as in Art. 6; and since by LEMMA VI, or rather by Art. 24, A is indefinitely diminutive, we have, by collecting homogeneous quantities

$$L + L' = 180^\circ$$

But A B, A D being ultimately *not indefinitely great*, it might easily be shown from Euclid that $L = L'$, and $\therefore$ A B = A D ultimately, (see Art. 6) and the intermediate arc is equal to either of them.

OTHERWISE,



If we refer the curve to its axis, A a, B b being ordinates, &c. as in the annexed diagram. Then, by Euclid, we have

$$A D^2 = A B^2 + B D^2 + 2 B D . B d$$

$$\therefore \frac{A D^2}{A B^2} = 1 + B D . \frac{B D + 2 B d}{A B^2}$$

Now, since by Art. 24 or LEMMA VI, the $\angle$ B A D is indefinitely less than either of the angles B or D, $\therefore$ B D is indefinite compared with A B or A D. Hence L being the limit of $\frac{A D}{A B}$ and l its variable part, if we extract the root of both sides of the equation and compare *homogeneous* terms, we get,

$$L = 1 \text{ or \&c. \&c.}$$

26. Having thus demonstrated that *the limiting Ratio of the chord, arc and tangent, is a ratio of equality, when the secant cuts the chord and tangent at FINITE angles,* we must again digress from the main object of this work, to take up the subject of Article 17. By thus deriving the limits of the ratios of the finite differences of functions and their variables, directly from the LEMMAS of this Section, and giving to such limits a convenient algorithm or notation, we shall not only clear up the doctrine of limits by numerous examples, but also prepare the way for understanding the abstruser parts of the Principia. This has been before observed.

Required to find the Limit of the Finite Differences of the sine of a circular arc and of the arc itself, or the Ratio of their Differentials.

Let x be the arc, and Δ x its finite variable increment. Then L being the limit required and L + l the variable ratio, we have

$$L + l = \frac{\Delta \sin. x}{\Delta x} = \frac{\sin. (x + \Delta x) - \sin. x}{\Delta x}$$

$$= \frac{\sin. x. \cos. (\Delta x) + \cos. x. \sin. (\Delta x) - \sin. x}{\Delta x}$$

$$= \cos. x. \frac{\sin. (\Delta x)}{\Delta x} + \frac{\sin. x. \cos. \Delta x}{\Delta x} - \frac{\sin. x}{\Delta x}$$

Now by LEMMA VII, as demonstrated in the preceding Article, the limit of $\frac{\sin. \Delta x}{\Delta x}$ is 1, and $\frac{\cos. (\Delta x)}{\Delta x}, - \frac{\sin. x}{\Delta x}$ have no *definite* limits.

Consequently putting

$$\cos. x. \frac{\sin. (\Delta x)}{\Delta x} = \cos. x + l',$$

we have

$$L + l = \cos. x + l' + \frac{\sin. x. \cos. \Delta x}{\Delta x} - \frac{\sin. x}{\Delta x}.$$

and equating homogeneous terms

$$L = \cos. x$$

or adopting the differential symbols

$$\left.\begin{array}{l} \dfrac{d. \sin. x}{d x} = \cos x \\ \text{or} \\ d \sin. x = d x. \cos. x \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

27. Hence and from the rules for the differentiation of algebraic, exponential, &c. functions, we can differentiate all other circular functions of one variable, viz. cosines, tangents, cotangents, secants, &c.

Thus,

$$\frac{d \sin. \left(\frac{\pi}{2} - x\right)}{d. \left(\frac{\pi}{2} - x\right)} = \cos. \left(\frac{\pi}{2} - x\right) = \sin. x$$

or

$$\frac{d. \cos. x}{- d x} = \sin. x$$

or

$$\left.\begin{array}{l} \dfrac{d. \cos. x}{d x} = - \sin. x \\ \text{or} \\ d. \cos. x = - d x. \sin. x \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (b)$$

Again, since for radius 1, which is generally used as being the most simple,

$$1 + \tan.^2 x = \sec.^2 x = \frac{1}{\cos.^2 x}$$

$$\therefore 2 \tan. x. d. \tan. x = d. \frac{1}{\cos.^2 x} = \frac{- 2 \cos. x. \; d. \cos. x}{\cos.^4 x}$$

See 12 (d). Hence and from (b) immediately above, we have

$$\tan. x. \; d. \; \tan. x = \frac{d x. \sin. x}{\cos.^3 x}$$

$$\therefore d. \tan. x = d x. \frac{1}{\cos.^2 x} \quad . \; . \; . \; . \; . \; . \quad (c)$$

Again,

$$\cot. x = \frac{1}{\tan. x}$$

Therefore,

$$\text{d. cot. x} = \text{d.} \frac{1}{\text{tan. x}} = \frac{-\text{d. tan. x}}{\text{tan.}^2 \text{x}} \quad (12.\ \text{d})$$

$$= \frac{-\text{d x}}{\text{tan.}^2 \text{x. cos.}^2 \text{x}} = \frac{-\text{d x}}{\text{sin.}^2 \text{x}} \quad \ldots \ldots \quad (\text{d})$$

Again,

$$\text{sec. x} = \frac{1}{\text{cos. x}}$$

$$\therefore \text{d. sec. x} = \text{d.} \frac{1}{\text{cos. x}} = \frac{-\text{d cos. x}}{\text{cos.}^2 \text{x}} \quad (12.\ \text{d})$$

$$= \frac{\text{d x. sin. x}}{\text{cos.}^2 \text{x}} \quad \ldots \ldots \ldots \quad (\text{e})$$

and lastly since cosec. x $=$ sec. $\left(\frac{\pi}{2} - \text{x}\right)$

we have

$$\text{d. cosec. x} = \text{d. sec.} \left(\frac{\pi}{2} - \text{x}\right) = \frac{\text{d.} \left(\frac{\pi}{2} - \text{x}\right) \text{sin.} \left(\frac{\pi}{2} - \text{x}\right)}{\text{cos.}^2 \left(\frac{\pi}{2} - \text{x}\right)}$$

$$= \frac{-\text{d x. cos. x}}{\text{sin.}^2 \text{x}} \quad \ldots \ldots \ldots \ldots \quad (\text{f})$$

Any function of sines, cosines, &c. may hence be differentiated.

28. In articles 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26 and 27, are to be found forms for the differentiation of any function of one variable, whether it be algebraic, exponential, logarithmic, or circular.

In those Articles we have found in short, *the limit of the* ratio of the first difference of a function, and of the first difference of its variable. Now suppose in this first difference of the function, the variable x should be increased again by Δ x, then taking the difference between the first difference and what it becomes when x is thus increased, we have the difference of the first difference of a function, or the second difference of a function, and so on through all the orders of differences, making Δ x always the same, merely for the sake of simplicity. Thus,

$$\Delta (\text{x}^3) = (\text{x} + \Delta \text{x})^3 - \text{x}^3$$

$$= 3 \text{x}^2 \Delta \text{x} + 3 \text{x} \Delta \text{x}^2 + \Delta \text{x}^3$$

and $\Delta^2 (\text{x})^3 = 3 (\text{x} + \Delta \text{x})^2 \Delta \text{x} + 3 (\text{x} + \Delta \text{x}) \Delta \text{x}^2 + \Delta \text{x}^3 - 3 \text{x}^2 \Delta \text{x}$

$$- 3 \text{x} \Delta \text{x}^2 - \Delta \text{x}^3$$

$$= 3.\ 2 \text{x} \Delta \text{x}^2 + 3 \Delta \text{x}^3$$

denoting by Δ^2 the second difference.

Hence,

$$\frac{\Delta^2 (x^3)}{\Delta x^2} = 3.\ 2.\ x + 3 \Delta x$$

and if the limiting ratio of $\Delta^2 (x^3)$ and Δx^2, or the ratio of the second differential of x^3, and the square of the differential of its variable x, be required, we should have

$$L + l = 3.\ 2.\ x + 3 \Delta x$$

and equating homogeneous terms

$$\frac{d^2 (x^3)}{d x^2} = L = 3.\ 2.\ x$$

In a word, without considering the difference, we may obtain the second, third, &c. differentials $d^2 u$, $d^3 u$, &c. of any function u of x immediately, if we observe that $\frac{d u}{d x}$ is always a function itself of x, and make d x constant. For example, let

$$u = a x^n + b x^m + \&c.$$

Then, from Art. 13. we have

$$\frac{d u}{d x} = n a x^{n-1} + m b x^{m-1} + \&c.$$

$$\frac{d. \left(\frac{d u}{d x}\right)}{d x} = \frac{d (d u)}{d x^2} = \frac{d^2 u}{d x^2} \text{ (by notation)}$$

$$= n. (n - 1) a x^{n-2} + m (m - 1) b x^{m-2} + \&c.$$

Similarly,

$$\frac{d^3 u}{d x^3} = n. (n - 1). (n - 2) a x^{n-3} + \&c.$$

$$\&c. = \&c.$$

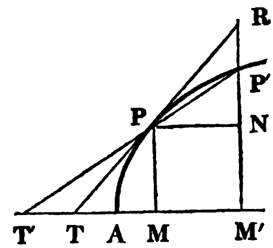
Having thus explained the method of ascertaining the limits of the ratios of all orders of finite differences of a function, and the corresponding powers of the invariable first difference of the variable, or the ratios of the differentials of all orders of a function, and of the corresponding power of the first differential of its variable, we proceed to explain the use of these limiting ratios, or ratios of differentials, by the following

APPLICATIONS

OF THE

DIFFERENTIAL CALCULUS.

29. *Let it be required to draw a tangent to a given curve at any given point of it.*



Let P be the given point, and A M being the axis of the curve, let P M = y, A M = x be the ordinate and abscissa. Also let P' be any other point; draw P N meeting the ordinate P' M' in N, and join P P'. Now let T P R meeting M' P' and M A in R and T be the tangent required.

Then since by similar triangles

$$P'N : PN :: PM : MT'$$

$$\therefore MT' = MT + TT' = y . \frac{\Delta x}{\Delta y}$$

Now y being supposed, as it always is in curves, a function of x, we have seen that whether that function be algebraic, exponential, &c.

$\frac{\Delta x}{\Delta y}$ in the limit, or $\frac{dx}{dy}$ is always a *definite function* of x. Hence putting

$$\frac{\Delta x}{\Delta y} = \frac{dx}{dy} + 1$$

we have

$$MT + TT' = y\left(\frac{dx}{dy} + 1\right)$$

and equating *homogeneous* terms,

$$MT = \frac{y\,dx}{dy} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (e)$$

which being found from the equation to the curve, the point T will be known, and therefore the position of the tangent P T. M T is called the subtangent.

Ex. 1. In the common parabola,

$$y^2 = ax$$

Therefore,

$$\frac{d\,x}{d\,y} = \frac{2\,y}{a}$$

and

$$M\,T = \frac{2\,y^2}{a} = 2\,x$$

or the subtangent M T is equal to twice the abscissa.

Ex. 2. In the ellipse,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

and it will be found by differentiating, &c. that

$$M\,T = \frac{-(a^2 - x^2)}{x}$$

Ex. 3. In the logarithmic curve,

$$y = a^x$$

$$\therefore \frac{d\,y}{d\,x} = l.\ a \times y \qquad \text{(see 17.)}$$

$$\therefore M\,T = \frac{1}{l\,a}$$

which is therefore the same for all points.

The above method of deducing the expression for the subtangent is strictly logical, and obviates at once the objections of Bishop Berkeley relative to the compensation of errors in the denominator. The fact is, these supposed errors being different in their very essence or nature from the other quantities with which they are connected, must in their aggregate be equal to nothing, as it has been shown in Art. 6. This ingenious critic calls P′ R = z; then, says he, (see fig. above)

$$M\,T = \frac{y.\,d\,x}{d\,y + z} \text{ accurately;}$$

whereas it ought to have been

$$M\,T = \frac{y\,\Delta\,x}{\Delta\,y + z} = \frac{y}{\frac{\Delta\,y}{\Delta x} + \frac{z}{\Delta\,x}}$$

the finite differences being here considered. Now in the limit, $\frac{\Delta\,y}{\Delta\,x}$ becomes a definite function of x represented by $\frac{d\,y}{d\,x}$. Consequently if l be put for the variable part of $\frac{\Delta\,y}{\Delta\,x}$, we have

$$M\,T = \frac{y}{\frac{d\,y}{d\,x} + 1 + \frac{z}{\Delta\,x}}$$

and it is evident from LEMMA VII and Art. 25, that z is indefinite compared with Δ x. $\therefore \frac{z}{\Delta\,x}$ is indefinite compared with M T, $\frac{d\,y}{d\,x}$, and y; and 1 is also so; hence

$$M\,T.\,\frac{d\,y}{d\,x} + \left(1 + \frac{z}{\Delta\,x}\right) M\,T = y$$

gives

$$M\,T = \frac{y.\,d\,x}{d\,y},\ \text{and}\ 1 + \frac{z}{\Delta\,x} = 0$$

which proves generally for all curves, what Berkeley established in the case of the common parabola; and at the same time demonstrates, as had been already done by using T T′ instead of P′ R, incontestably the accuracy of the equation for the subtangent.

30. If it were required to draw a tangent to any point of a curve, referred to a center by a *radius-vector* ϱ and the $\angle\ \theta$ which ϱ describes by revolving round the fixed point, instead of the rectangular coordinates x, y; then the mode of getting the subtangent will be somewhat different.

Supposing x to originate in this center, it is plain that

$$\left.\begin{aligned} x &= \varrho\ \cos.\ \theta \\ y &= \varrho\ \sin.\ \theta \end{aligned}\right\}$$

and substituting for x, y, d x, d y, hence derived in the expression (29. e.) we have

$$M\,T = \varrho\ \sin.\ \theta \times \frac{d\,\varrho\ \cos.\ \theta - \varrho\ d\,\theta\ \sin.\ \theta}{d\,\varrho\ \sin.\ \theta + \varrho\ d\,\theta\ \cos.\ \theta} \quad .\ .\ .\ .\ \text{(f)}$$

Ex. In the parabola

$$\varrho = \frac{2\,a}{1 - \cos.\ \theta},$$

where a is the distance between the focus and vertex, or the value of ϱ at the vertex. Then substituting we get, after proper reductions

$$M\,T = 2\,a \times \frac{1 + \cos.\ \theta}{1 - \cos.\ \theta}$$

and the distance from the focus to the extremity of the subtangent is

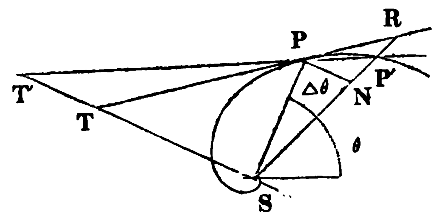
$$M\,T - \varrho\ \cos.\ \theta = 2\,a\left(\frac{1 + \cos.\ \theta}{1 - \cos.\ \theta} - \frac{\cos.\ \theta}{1 - \cos.\ \theta}\right)$$

$$= \frac{2a}{1 - \cos.\theta} = \varrho,$$

as is well known.

30. a. The expression (f) being too complicated in practice, the following one may be substituted for it.

Let P T be a tangent to the curve, referred to the center S, at the point P, meeting S T drawn at right angles to S P, in T; and let P′ be any other point. Join P P′ and produce it to T′, and let T P be produced to meet S P′ produced in R, &c. Then drawing P N parallel to S T, we have



$$S T' = S T + T T' = \frac{P N}{N P'} \times S P'$$

But

$$P N = \varrho \tan.\Delta\theta,\ \ S P' = \varrho + \Delta\varrho$$

and

$$N P' = S P' - S N = \varrho + \Delta\varrho - \frac{\varrho}{\cos.(\Delta\theta)}.$$

Therefore, substituting and equating homogeneous terms, after having applied Lemma VII to ascertain their limits, we get

$$S T = \frac{\varrho^2 d\theta}{d\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

Ex. 1. In the spiral of Archimedes we have

$$\varrho = a\theta;$$

$$\therefore S T = \frac{\varrho^2}{a}.$$

Ex. 2. In the hyperbolic spiral

$$\varrho = \frac{a}{\theta};$$

$$\therefore S T = -a$$

31. It is sometimes useful to know the angle between the tangent and axis.

$$\text{Tan. } T = \frac{P M}{M T} = \frac{d y}{d x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

See fig. to Art. 29.

Again, in fig. Art. 30 a.

$$\text{Tan. T} = \frac{\text{S P}}{\text{S T}} = \frac{d\,\varrho}{\varrho\, d\,\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (k)$$

32. It is frequently of great use, in the theory of curves and in many other collateral subjects, to be able to expand or develope any given function of a variable into an infinite series, proceeding according to the powers of that variable. We have already seen one use of such developments in Art. 17. This may be effected in a general manner by aid of successive differentiations, as follows.

If $u = f(x)$ where $f(x)$ means any function of x, or any expression involving x and constants; then, as it has been seen,

$$d\,u = u'\,d\,x$$

(u' being a new function of x)

Similarly

$$\begin{aligned} d\,u' &= u''\,d\,x \\ d\,u'' &= u'''\,d\,x \\ \&c. &= \&c. \end{aligned}$$

But

$$\begin{aligned} d\,u' &= d.\left(\frac{d\,u}{d\,x}\right) = \frac{d^2 u \times d\,x - d^2 x \times d\,u}{d\,x^2}, \quad (6\,k) \\ \&c. &= \&c. \end{aligned}$$

denoting d. (d u), d. (d x) by $d^2 u$, $d^2 x$, and $(d\,x)^2$ by $d\,x^2$, according to the received notation;

Or, (to abridge these expressions) supposing $d\,x$ constant, and $\therefore d^2 x = 0$,

$$d\,u' = \frac{d^2 u}{d\,x}$$

$$\therefore \left.\begin{aligned} \frac{d\,u}{d\,x} &= u' \\ \frac{d^2 u}{d\,x^2} &= u'' \\ \frac{d^3 u}{d\,x^3} &= u''' \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

$$\&c. = \&c.$$

which give the various orders of fluxions required.

Ex. 1. Let $u = x^n$

Then

$$\frac{d\,u}{d\,x} = n\,x^{n-1}$$

$$\frac{d^2 u}{d\,x^2} = n.\;(n-1)\,x^{n-2}$$

$$\frac{d^3 u}{d x^3} = n. (n - 1). (n - 2) x^{n-3}$$

&c. = &c.

$$\frac{d^n u}{d x^n} = n. (n - 1). (n - 2) \dots \dots 3. 2. 1.$$

Ex. 2. Let $u = A + B x + C x^2 + D x^3 + E x^4 + \&c.$

Then,

$$\frac{d u}{d x} = B + 2 C x + 3 D x^2 + 4 E x^3 + \&c.$$

$$\frac{d^2 u}{d x^2} = 2 C + 2. 3 D x + 3. 4 E x^2 + \&c.$$

$$\frac{d^3 u}{d x^3} = 2. 3 D + 2. 3. 4 E x + \&c.$$

&c. = &c.

Hence, if *u* be known, and the coefficients A, B, C, D, &c. be unknown, the latter may be found; for if U, U′, U″, U‴, &c. denote the values of u, $\frac{d u}{d x}, \frac{d^2 u}{d x^2}, \frac{d^3 u}{d x^3}$, &c.

when $x = 0$, then

$$A = U, B = U', C = \frac{1}{2} \cdot U'', D = \frac{1}{2. 3.} \cdot U''', E = \frac{1}{2. 3. 4} \cdot U'''',$$

&c. = &c.

and by substitution,

$$u = U + U' x + U'' \frac{x^2}{2} + U''' \frac{x^3}{2. 3} + \&c. \quad \dots \dots \quad (b)$$

This method of discovering the coefficients is named (after its inventor),

MACLAURIN'S THEOREM.

The uses of this Theorem in the expansion of functions into series are many and obvious.

For instance, let it be required to develope sin. x, or cos. x, or tan. x, or l. (1 + x) into series according to the powers of x. Here

$$u = \sin. x, \text{ or } = \cos. x, \text{ or } = \tan. x, \text{ or } = l. (1 + x),$$

$$\therefore \frac{d u}{d x} = \cos. x, \text{ or } = - \sin. x, \text{ or } = \frac{1}{\cos.^2 x}, \text{ or } = \frac{1}{1 + x}$$

$$\frac{d^2 u}{d x^2} = - \sin. x, \text{ or } = - \cos. x, \text{ or } = \frac{2 \sin. x}{\cos.^3 x}, \text{ or } = - \frac{1}{(1 + x)^2}$$

$$\frac{d^3 u}{d x^3} = -\cos. x, \text{ or } = \sin. x, \text{ or } = \frac{2 + 4 \sin.^2 x}{\cos.^4 x}, \text{ or } = \frac{2}{(1 + x)^3}$$

&c. = &c.

$$\begin{aligned} \therefore\ U &= 0, \quad \text{or} = 1, \quad \text{or} = 0, \text{ or} = 0 \\ U' &= 1, \quad \text{or} = 0, \quad \text{or} = 1, \text{ or} = 1 \\ U'' &= 0, \quad \text{or} = -1, \text{ or} = 0, \text{ or} = -1 \\ U''' &= -1, \text{ or} = 0, \quad \text{or} = 2, \text{ or} = 2 \\ \&c. &= \&c. \end{aligned}$$

Hence

$$\sin. x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c.$$

$$\cos. x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \&c.$$

$$\tan. x = x + \frac{x^3}{3} + \frac{2 x^5}{3.5} + \frac{17 x^7}{3^2 5.7} + \&c.$$

$$l. (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

Hence may also be derived

TAYLOR'S THEOREM.

For let

$$f(x) = A + B x + C x^2 + D x^3 + E x^4 + \&c.$$

Then

$$\begin{aligned} f(x + h) &= A + B.(x + h) + C.(x + h)^2 + D.(x + h)^3 + \&c. \\ &= A + B x + C x^2 + D x^3 + \&c. \\ &\quad + (B + 2 C x + 3 D x^2) h \\ &\quad + (C + 3 D x + 6 E x^2) h^2 \\ &\quad + (D + 4 E x + 10 F x^2) h^3 \\ &\quad + \&c. \\ &= f(x) + \frac{d. f(x)}{d x} h + \frac{d.^2 f(x)}{d x^2} \cdot \frac{h^2}{2} \\ &\quad + \frac{d^3 f(x)}{d x^3} \cdot \frac{h^3}{2.3} + \&c. \quad \ldots \ldots \ldots \quad (c) \end{aligned}$$

the theorem in question, which is also of use in the expansion of series.

For the extension of these theorems to functions of two or more *variables*, and for the still more effective theorems of Lagrange and Laplace, the reader is referred to the elaborate work of Lacroix. 4to.

Having shown the method of finding the differentials of any quanti-

ties, and moreover, entered in a small degree upon the practical application of such differentials, we shall continue for a short space to explain their farther utility.

33. *To find the* MAXIMA *and* MINIMA *of quantities.*

If a quantity increase to a certain magnitude and then decrease, the state between its increase and decrease is its *maximum*. If it decrease to a certain limit, and then increase, the intermediate state is its *minimum*. Now it is evident that in the change from increasing to decreasing, or *vice versa*, which the quantity undergoes, its differential must have changed signs from positive to negative, or *vice versa*, and therefore (since moreover this change is *continued*) have passed through *zero*. Hence *When a quantity is a* MAXIMUM *or* MINIMUM, *its differential* $= 0$. . . (a)

Since a quantity may have several different maxima and minima, (as for instance the ordinate of an undulating kind of curve) it is useful to have some means of distinguishing between them.

34. *To distinguish between* MAXIMA *and* MINIMA.

LEMMA. To show that in Taylor's Theorem (32. c.) any one term can be rendered greater than the sum of the succeeding ones, supposing the coefficients of the powers of h to be finite.

Let $Q\,h^{n-1}$ be any term of the theorem, and P the greatest coefficient of the succeeding terms. Then, supposing h less than unity,

$$P\,h^n\,(1 + h + h^2 + \ldots \text{ in infin.}) = P\,h^n \times \frac{1}{1-h}$$

is greater than the sum (S) of the succeeding terms. But supposing h to decrease *in infin.*

$$P\,h.^n \frac{1}{1-h} = P\,h^n \textit{ ultimately}. \quad \text{Hence } \textit{ultimately}$$

$$P\,h^n > S\,.$$

Now

$$Q\,h^{n-1} : P\,h^n :: Q : P\,h,$$

and since Q and P are finite, and h infinitely small; therefore Q is $>$ P h, Hence $Q\,h^{n-1}$ is $>$ $P\,h^n$, and a fortiori $>$ S.

Having established this point, let

$$u = f(x)$$

be the function whose *maxima* and *minima* are to be determined; also when u = max. or min. let x = a. Then by Taylor's Theorem

$$f(a - h) = f(a) - \frac{d\,u}{d\,a}\,h + \frac{d^2 u}{d\,a^2}.\,\frac{h^2}{1.2} - \frac{d^3 u}{d\,a^3}.\,\frac{h^3}{2.3} + \&c.$$

and

$$f(a+h) = f(a) + \frac{du}{da}h + \frac{d^2u}{da^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{da^3}\cdot\frac{h^3}{2.3} + \&c.$$

and since by the LEMMA, the sign of each term is the sign of the sum of that and the subsequent terms,

$$\therefore f(a-h) = f(a) - \frac{du}{da}.\ M$$

$$f(a+h) = f(a) + \frac{du}{da}.\ N$$

Now since f (a) = max. or min. f (a) is > or < than both f (a — h) and f (a + h), which cannot be unless

$$\frac{du}{da} = 0.$$

Hence

$$\left.\begin{aligned} f(a-h) &= f(a) + \frac{d^2u}{da^2}.\ M' \\ f(a+h) &= f(a) + \frac{d^2u}{da^2}.\ N' \end{aligned}\right\}$$

and f (a) is *max.* or *min.* or *neither*, according as f (a) is >, < or = to both f (a — h) and f (a + h), or according as

$\frac{d^2u}{da^2}$ is *negative*, *positive*, or *zero*.....

If it be *zero* as well as $\frac{du}{da}$, we have

$$\left.\begin{aligned} f(a-h) &= f(a) - \frac{d^3u}{da^3}.\ M'' \\ f(a+h) &= f(a) + \frac{d^3u}{da^3}.\ N'' \end{aligned}\right\}$$

and f (a) cannot = max. or min. unless

$$\frac{d^3u}{da^3} = 0,$$

which being the case we have

$$\left.\begin{aligned} f(a-h) &= fa + \frac{d^4u}{da^4}.\ M''' \\ f(a+h) &= fa + \frac{d^4u}{da^4}.\ N''' \end{aligned}\right\}$$

and as before,

f (a) is *max.* or *min.* or *neither*, according as $\frac{d^4 u}{d a^4}$ is *negative, positive*, or *zero*, and so on continually.

Hence the following *criterion.*

If in u = f (x), $\frac{d u}{d x} = 0$, *the resulting value of* x *shall give u* = MAX. or MIN. or NEITHER, *according as* $\frac{d^2 u}{d x^2}$ *is negative, positive*, or *zero.*

If $\frac{d u}{d x} = 0$, $\frac{d^2 u}{d x^2} = 0$, and $\frac{d^3 u}{d x^3} = 0$, *then the resulting value of u shall be a* MAX., MIN. *or* NEITHER *according as* $\frac{d^4 u}{d x^4}$ *is* NEGATIVE, POSITIVE, *or* ZERO; *and so on continually.*

Ex. 1. *To find the* MAX. *and* MIN. *of the ordinate of a common parabola.*

$$y = \sqrt{a x}$$

$$\therefore \frac{d y}{d x} = \frac{1}{2} \cdot \frac{\sqrt{a}}{\sqrt{x}}$$

which cannot = 0, unless x = α.

Hence the parabola has no maxima or minima ordinates.

Ex. 2. *To find the* MAXIMA *and* MINIMA *of y in the equation*

$$y^2 - 2 a x y + x^2 = b^2.$$

Here

$$\frac{d y}{d x} = \frac{a y - x}{y - a x}, \frac{d^2 y}{d x^2} = \frac{2 a \frac{d y}{d x} - \left(\frac{d y}{d x}\right)^2 - 1}{y - a x}$$

and putting $\frac{d y}{d x} = 0$, we get

$$x = \frac{\pm a b}{\sqrt{(1 - a^2)}}, y = \frac{\pm b}{\sqrt{(1 - a^2)}}, \frac{d}{d x^2} = \frac{\mp 1}{b \sqrt{(1 - a^2)}}$$

which indicate and determine both a maximum and a minimum.

Ex. 3. *To divide* a *in such a manner that the product of the* m^{th} *power of the one part, and the* n^{th} *power of the other shall be a maximum.*

Let x be one part, then a — x = the other, and by the question

$$u = x^m. (a - x)^n = \max.$$

$$\therefore \frac{d u}{d x} = x^{m-1}. (a - x)^{n-1} \times (m a - x. \overline{m + n})$$

and

$$\frac{d^2 u}{d x^2} = x^{m-2} (a - x)^{n-2} \times (\overline{m + n - 1}. \overline{m + n}. x^2 - \&c.)$$

Put $\frac{d u}{d x} = 0$; then

$$x = 0, \text{ or } x = a, \text{ or } x = \frac{m a}{m + n},$$

the two former of which when *m* and *n* are *even* numbers give *minima*, and the last the required *maximum*.

Ex. 4. Let $u = x^{\frac{1}{x}}$.

Here

$\frac{d u}{d x} = u. \frac{1 - l. x}{x^2} = 0, \therefore l. x = 1$, and $x = e$ the *hyperbolic* base $= 2.71828$, &c.

Innumerable other examples occur in researches in the doctrine of curves, optics, astronomy, and in short, every branch of both abstract and applied mathematics. Enough has been said, however, fully to demonstrate the general principle, when applied to functions of *one independent variable only*.

For the MAXIMA and MINIMA of functions of two or more variables, see *Lacroix*, 4to.

35. If in the expression (30 a. g) S T should be finite when ϱ is infinite, then the corresponding tangent is called an *Asymptote* to the curve, and since ϱ and this Asymptote are both infinite they are parallel. Hence

To find the Asymptotes to a curve,

In $S T = \varrho^2 \frac{d \theta}{d \varrho}$, make $\varrho = \alpha$, then each *finite* value of S T gives an Asymptote; which may be drawn, by finding from the equation to the curve the values of θ for $\varrho = \alpha$, (which will determine the positions of ϱ), then by drawing through S at right angles to ϱ, S T, S T′, S T″, &c. the several values of the subtangent of the asymptotes, and finally through T, T′, T″, &c. perpendiculars to S T, S T′, S T″, &c. These perpendiculars will be the asymptotes required.

Ex. In the hyperbola

$$\varrho = \frac{b^2}{a (1 - e \cos. \theta)}.$$

Here $\varrho = \alpha$, gives $1 - e \cos. \theta = 0, \quad \therefore \cos. \theta = \frac{1}{e}$

$\therefore \pm \theta = \angle$ whose cos. is $\frac{1}{e}$.

Also $S\,T = \frac{b^2}{a\,e\,\sin.\,\theta} = \frac{b^2}{a\sqrt{e^2-1}} = b$; whence it will be seen that the asymptotes are equally inclined (viz. by $\angle\,\theta$) to the axis, and pass through the center.

The expression (29. e) will also lead to the discovery and construction of *asymptotes*.

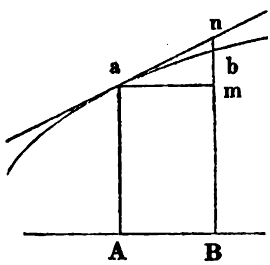
Since the tangent is the nearest straight line that can be drawn to the curve at the point of contact, it affords the means of ascertaining the inclination of the curve to any line given in position; also whether at any point the curve be *inflected*, or from *concave* become *convex* and vice versa; also whether at any point two or more branches of the curve meet, i. e. whether that *point* be *double*, *triple*, &c.

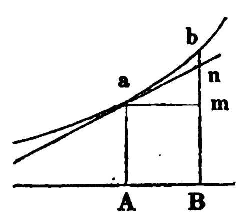
36. *To find the inclination of a curve at any point of it to a given line; find that of the tangent at that given point, which will be the inclination required.*

Hence if the inclination of the tangent to the axis of a curve be zero, the ordinate will then be a *maximum* or *minimum*; for then

$$\tan.\,T = \frac{d\,y}{d\,x} = 0 \qquad (31.\ h)$$

37. *To find the points of Inflexion of a curve.*





Let $y = f(x)$ be the equation to the curve a b; then A a, B b being any two ordinates, and a n a tangent at the point a, if we put $A\,a = y$, and $A\,B = h$, we get

$$A\,a = f\,x$$

$$B\,b = f(x+h) = y + \frac{d\,y}{d\,x}\,h + \frac{d^2\,y}{d\,x^2}\,\frac{h^2}{1.\,2} + \&c. \qquad (32.\ c)$$

$$= y + \frac{d\,y}{d\,x}\,h + \frac{d^2\,y}{d\,x^2}\,M \qquad (34)$$

But $B\,n = y + m\,n = y. + \frac{d\,y}{d\,x}.\,h$. Consequently B b is $<$ or $>$ B n according as $\frac{d^2\,y}{d\,x^2}$ is negative or positive, i. e. *the curve is concave or con-*

vex towards its axis according as $\frac{d^2 y}{d x^2}$ *is negative or positive.*

Hence also, since a quantity in passing from positive to negative, and *vice versa,* must become zero or infinity, *at a point of inflexion*

$$\frac{d^2 y}{d x^2} = 0 \text{ or } \infty$$

Ex. *In the Conchoid of Nicomedes*

$$x y = (a + y) \surd (b^2 - y^2)$$

which gives, by making d y constant,

$$\frac{d^2 x}{d y^2} = \frac{2 b^4 a - b^2 y^3 - 3 b^2 a y^2}{(b^2 y^3 - y^5) \surd (b^2 - y^2)}$$

and putting this = 0, and reducing, there results

$$y^3 + 3 a y^2 = 2 b^2 a$$

which will give y and then x.

These points of *inflexion* are those which the Theory of (34) indicates as belonging to neither *maxima* nor *minima;* and pursuing this subject still farther, it will be found, in like manner, that in some curves

$$\frac{d^4 y}{d x^4} = 0 \text{ or } \infty, \frac{d^6 y}{d x^6} = 0 \text{ or } \infty, \&c. = \&c.$$

also determine *Points* of *Inflexion.*

38. *To find* DOUBLE, TRIPLE, *&c. points of a curve.*

If the branches of the curve cut one another, there will evidently be as many tangents as branches, and consequently either of the expressions,

$$\text{Tan. T} = \frac{d y}{d x} \qquad (31. \text{h})$$

$$\text{M T} = \frac{y \, d x}{d y} \qquad (29. \text{e})$$

as derived from the equation of the curve, will have as many values as there are branches, and thus the nature and position of the point will be ascertained.

If the branches of the curve touch, then the tangents coincide, and the method of discovering such *multiple points* becomes too intricate to be introduced in a brief sketch like the present. For the entire Theory of Curves the reader is referred to Cramer's express treatise on that subject, or to Lacroix's Different. and Integ. Calculus, 4to. edit.

39. We once more return to the text, and resume our comments. We pass by LEMMA VIII as containing no difficulty which has not been already explained.

As similar figures and their properties are required for the demonstra-

tion of Lemma IX, we shall now use Lemma VII in establishing Lemma V, and shall thence proceed to show what figures are similar and how to construct them.

According to Newton's notion of similar curvilinear figures, we may define *two curvilinear figures to be similar when any rectilinear polygon being inscribed in one of them, a rectilinear polygon similar to the former, may always be inscribed in the other.*

Hence, increasing the number of the sides of the polygons, and diminishing their lengths indefinitely, the lengths and areas of the curvilinear figures will be the limits by Lemmas VII and III, of those of the rectilinear polygons, and we shall, therefore, have by Euclid these lengths and areas in direct and duplicate proportions of the homologous sides respectively.

40. *To construct curves similar to given ones.*

If y, x be the ordinate and abscissa, and x′ the corresponding abscissa of the required curve, we have

$$x : y :: x' : \frac{y}{x} \times x' = y' \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the ordinate of the required curve, which gives that point in it which corresponds to the point in the given curve whose coordinates are x, y; and in the same manner may as many other points as we please be determined.

In such curves, however, as admit a practical or mechanical construction, it will frequently be sufficient to determine but one or two values of y′.

Ex. 1. In the circle let x, measured along the diameter from its extremity, be r (the radius); then y = r, and we have

$$y' = \frac{y}{x} \times x' = x'$$

where x′ may be of any magnitude whatever. Hence, *all semicircles, and therefore circles, are similar figures.*

Ex. 2. In a circular arc (2 α) let x be measured along the chord (2 b), and suppose $x = r \sin. \alpha$; then $y = r . \text{vers.} \alpha$

$$y' = \frac{\text{vers.} \alpha}{\sin. \alpha} \times x'$$

which gives the greatest ordinate to any semichord as an abscissa, of the required arc, and thence since

$$y' = r' - \sqrt{r'^2 - x'^2}$$

it will be easy to find the radius r′ and centre, and to describe the arc required.

But since

$$\frac{y'}{x'} = \frac{r' \text{ vers. } \alpha'}{r' \sin. \alpha'} = \frac{\text{vers. } \alpha'}{\sin. \alpha'} = \frac{\text{vers. } \alpha}{\sin. \alpha}$$

therefore

$$\frac{1 - \cos. \alpha}{\sin. \alpha} = \frac{2 \sin.^2 \frac{\alpha}{2}}{2 \cos. \frac{\alpha}{2} \sin. \frac{\alpha}{2}} = \frac{1 - \cos. \alpha'}{\sin. \alpha'} = \frac{2 \sin.^2 \frac{\alpha'}{2}}{2 \cos. \frac{\alpha'}{2} \sin. \frac{\alpha'}{2}}$$

or

$$\tan. \frac{\alpha}{2} = \tan. \frac{\alpha'}{2},$$

and

$$\therefore \alpha = \alpha'$$

which accords with Euclid, and shows *that similar arcs of circles subtend equal angles.*

Ex. 3. *Given an arc of a parabola, whose latus-rectum is* p, *to find a similar one, whose latus-rectum shall be* p′.

In the first place, since the arc is given, the coordinates at its extremities are; whence may be determined its axis and vertex; and by the usual mode of describing the parabola it may be completed to the vertex. Now, since

$$y^2 = p\,x$$

x, x′ being measured along the axis, and when

$$x = \frac{p}{4}, \; y = \frac{p}{2}$$

$$\therefore y' = \frac{y}{x} . x' = \frac{p}{y} . x' = 2\,x'$$

which shows *that all semi-parabolas, and therefore parabolas, are similar figures.* Hence, having described upon the axis of the given parabola, any other having the same vertex, the arc of this latter intercepted between the points whose coordinates correspond to those of the extremities of the given arc will be the arc required.

Ex. 4. In the ellipse whose semi-diameters are a, b, if x be measured along the axis, when x = a, y = b. Hence

$$y' = \frac{b}{a} . x'$$

and x′ or the *semi-axis major* being assumed any whatever, this value of y′ will give the *semi-axis minor*, whence the ellipse may be described.

This being accomplished, let (α, β) (α', β') be the coordinates at the

extremities of any *given arc* of the given ellipse, then the similar one of the ellipse described will be that intercepted between the points whose coordinates, (x', y') (x'', y'') are given by

$$\left.\begin{array}{l}\alpha : \beta : : x' : y' \\ \alpha' : \beta' : : x'' : y''\end{array}\right\} \text{ and } \begin{array}{l} y' = \frac{b'}{a'} \sqrt{(2\, a'\, x' - x'^2)} \\ y'' = \frac{b''}{a''} \sqrt{(2\, a''\, x'' - x''^2)}\end{array}$$

In like manner it may be found, that

All cycloids are similar.

Epicycloids are so, when the radii of their wheels $\propto$ *radii of the spheres.*

Catenaries are similar when the bases $\propto$ *tensions, &c. &c.*

40. If it were required to describe the curve A c b (fig. to LEMMA VII) not only similar to A C B, but also such that its chord should be of the given length (c); then having found, as in the last example, the coordinates (x', y') (x'', y'') in terms of the assumed value of the abscissa (as a' in Ex. 4), and (α, β), (α', β') the coordinates at the extremities of the given arc, we have

$$c = \sqrt{(x' - x'')^2 + (y' - y'')^2} = f(\alpha')$$

a function of a': whence a' may be found.

Ex. In the case of a parabola whose equation is $y^2 = a\, x$, it will be found that ($y'^2 = a'\, x'$ being the equation of the required parabola)

$$c = \frac{a'}{a^2} \cdot (\beta - \beta') \sqrt{(\beta + \beta')^2 + \alpha^2},$$

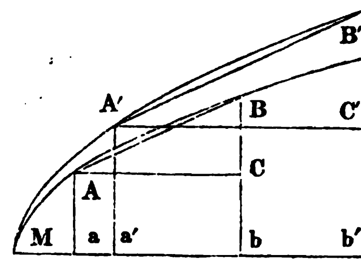
whence (a') is known, or the latus-rectum of the required parabola is so determined, that the arc similar to the given one shall have a chord = c.

41. It is also assumed in the construction both to LEMMA VII and LEMMA IX, that, *If in similar figures, originating in the same point, the chords or axes coincide, the tangents at that origin will coincide also.*

Since the chords A B, A b (fig. to LEMMA VII), the *parallel* secants B D, b d, and the tangents A D, A d are corresponding sides, each to each, to the similar figures, we have (by LEMMA V)

$$A\,B : B\,D : : A\,b : b\,d$$

and $\angle B = \angle b$. Consequently, by Euclid the $\angle B\,A\,D = \angle b\,A\,d$, or the tangents coincide.



To make this still clearer. Let M B, M B' be two similar curves, and A B, A' B' similar parts of them. Let fall from A, B, A', B', the ordinates A a, B b, A' a', B' b' cutting off the corresponding abscissæ M a, M b, M a', M b', and draw the chords A B, A' B'; also draw A C, A' C' at right angles to B b, B' C'.

Then, since (by LEMMA V)

$$\left.\begin{array}{l} \mathrm{M\,a : M\,b :: A\,a : B\,b} \\ \mathrm{M\,a' : M\,b' :: A'\,a' : B'\,b'} \end{array}\right\}$$

$$\therefore \left.\begin{array}{l} \mathrm{M\,a : a\,b :: A\,a : B\,C} \\ \mathrm{M\,a' : a'\,b' :: A'\,a' : B'\,C'} \end{array}\right\}$$

$$\therefore \left.\begin{array}{l} \mathrm{A\,C : B\,C :: M\,a : A\,a} \\ \mathrm{A'\,C' : B'\,C' :: M\,a' : A'\,a'} \end{array}\right\}$$

But

$$\mathrm{M\,a : A\,a :: M\,a' : A'\,a'}$$

$$\therefore \mathrm{A\,C : B\,C :: A'\,C' : B'\,C'}$$

and the $\angle$ C = $\angle$ C'

$\therefore$ the triangles A B C, A' B' C' are similar, and the $\angle$ B A C = $\angle$ B' A' C', i. e. A B is parallel to A' B'.

Hence if B, B' move up to A, the chords A B, A' B' shall *ultimately* be parallel, i. e. the tangents (see LEMMA III, Cor. 2 and 3, or LEMMA VI,) at A, A' are parallel.

Hence, if the chords coincide, as in fig. to LEMMA VII, the tangents coincide also.

The student is now prepared for the demonstration of the LEMMA. He will perceive that as B approaches A, new curves, or parts of curves, A c b similar to the parts A C B are supposed continually to be described, the point *b* also approaching *d*, which may not only be at a *finite* distance from A, but absolutely fixed. It is also apparent, that as the ratio between A B and A b decreases, the curve A c b approaches to the straight line A b as its limit.

42. LEMMA XI. The construction will be better understood when thus effected.

Take A e of any given magnitude and draw the ordinate e c meeting A C produced in c, and upon A c describe the curve A b c (see 39)

similar to A B C. Take $A d = A e \times \frac{A D}{A E}$ and erect the ordinate d b meeting A b c in b. Then, since A d, A e are the abscissæ corresponding to A D, A E, the ordinates d b, e c also correspond to the ordinates D B, E C, and by LEMMA V we have

$$d b : D B :: e c : E C :: A e : A E$$
$$:: A d : A C \text{ (by construction)}$$

and the $\angle D = \angle d$. Hence
b is in the straight line A B produced, &c. &c.

43. This LEMMA may be proved, without the aid of similar curves, as follows:

$$A B D = \frac{A D}{2} . (D F + F B)$$
$$= A D^2 . \frac{\tan. \alpha}{2} + \frac{A D . B F}{2}$$

and

$$A C E = A E^2 . \frac{\tan. \alpha}{2} + \frac{A E . C G}{2}$$

where $\alpha = \angle D A F$.

$$\therefore \frac{A B D}{A C E} = \frac{A D^2 . \tan. \alpha + A D . B F}{A E^2 . \tan. \alpha + A E . C G}$$

Now by LEMMA VII, since $\angle B A F$ is indefinite compared with F or B; therefore B F, C G are indefinite compared with A D or A E. Hence if L be the limit of $\frac{A B D}{A C E}$, and L + l its varying value, we have

$$L + l = \frac{A D^2 . \tan. \alpha + A D . B F}{A E^2 . \tan. \alpha + A E . C G}$$

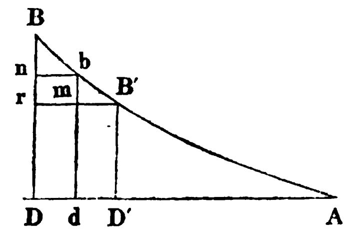
and multiplying by the denominator and equating homogeneous terms we get

$$L . A E^2 . \tan. \alpha = A D^2 . \tan. \alpha$$
$$\therefore \text{Limit of } \frac{A B D}{A C E} = \frac{A D^2}{A E^2}.$$

44. LEMMA X. "*Continually* increased or diminished." The word "*continually*" is here introduced for the same reason as "*continued curvature*" in LEMMA VI.

If the force, moreover, be not "*finite*," neither will its effects be; or the velocity, space described, and time will not admit of comparison.

45. Let the time A D be divided into several portions, such as D d, A b B being the *locus* of the extremities of the ordinates which D represent, the velocities acquired D B, d b, &c. Then upon these lines D d, &c. as bases, there being inscribed rectangles in the figure A D B, and when their number is increased and bases diminished indefinitely, their ultimate sum shall = the curvilinear area A B D (LEMMA III.) But each of these rectangles represents the space described in the time denoted by its base; for during an *instant* the velocity may be considered constant, and by mechanics we have for constant velocities S = T × V. Hence the area A B D represents the whole space described in the time A D.



In the same manner, A C E (see fig. LEMMA X) represents the time A E. But by LEMMA IX these areas are "ipso motus initio," as AD^2 and AE^2 Hence, in the very beginning of the motion, the spaces described are also in the duplicate ratio of the times.

46. Hence may be derived the differential expressions for the *space described*, *velocity acquired*, &c.

Let the velocity B D acquired in the time t (A D) be denoted by v, and the space described, by s.

Then, *ultimately*, we have

$$D d = d t, B n = d v,$$

and

$$D n b d = d s = D d \times d b = d t \times v.$$

Hence

$$v = \frac{d s}{d t}, \; d s = v \, d t, d t = \frac{d s}{v} \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

Again, if D d = d D′, the spaces described in these successive *instants*, are D b, D′ m, and therefore *ultimately* the fluxion of the space represented by the ultimate state of D′ m is b n r m or 2 b m B′. Hence

$$d (d s) = 2 \times b m B' \text{ ultimately},$$

and supposing B′ to move up to A, since in the limit at A, B′ coincides with A, and B′ m with A D, and therefore b m B′ or d (d s) represents the space described "in the very beginning of the motion."

Hence by the LEMMA,

$$d (d s) \propto 2 d t^2 \propto d t^2$$

or with the same accelerating force

$$d^2 s \propto d t^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (b)$$

With different accelerating forces $d^2 s$ must be proportionably increased or diminished, and $\therefore$ (see Wood's Mechanics)

$$d^2 s \propto F d t^2$$

Hence we have, after properly adjusting the units of force, &c.

$$\left.\begin{aligned} d^2 s &= F d t^2 \\ F &= \frac{d^2 s}{d t^2} \end{aligned}\right\} \quad \text{and} \therefore \quad \ldots\ldots\ldots \quad (c)$$

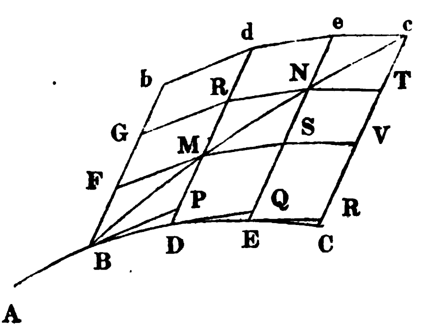
Hence also and by means of (a) considering d t constant,

$$F = \frac{d v}{d t},\ v d v = F d s \quad \ldots\ldots \quad (d)$$

all of which expressions will be of the utmost use in our subsequent comments.

47. Lemma X. Cor. I. To make this corollary intelligible it will be useful to prove the general principle, that

If a body, moving in a curve, be acted upon by any new accelerating force, the distance between the points at which it would arrive WITHOUT *and* WITH *the new force in the same time, or "error," is equal to the space that the new force, acting solely, would cause it to describe in that same time.*



Let a body move in the curve A B C, and when at B, let an additional force act upon it in the direction B b. Also let B D, D E, E C; B F, F G, G b be spaces that would be described in equal times by the body moving in the curve, and when moved by the sole action of the new force. Then draw tangents at the points B, D, E meeting D d, E e, C c, each parallel to B b, in P, Q, R. Also draw F M, G R, b d parallel to B P; M S, R N, d e parallel to D Q; and S V, N T, e c parallel to E R.

Now since the body at B is acted upon by forces which separately would cause it to move through B D, B F, or, when the number of the spaces is increased and their magnitude diminished *in infinitum*, through B P, B F in same time, therefore by LAW III, Cor. 1, when these forces act together, the body will move in that time through the diagonal up to M. In the same manner it may be shown to move from M to N, and from N to C in the succeeding times. Hence, if the number of the times be increased and their duration indefinitely diminished, the body will have moved through an indefinite number of points M, N, &c. up to C, describing a curve B C. Also since b d, d e, e c are each parallel to the tangents at B, D, E, or ultimately to the curve B D E C; ∴ b d e c ultimately assimilates itself to a curve equal and parallel to B D E C; moreover C c is parallel to B b. Hence C c is also equal to B b.

Hence, then, *The Error caused by any disturbing force acting upon a body moving in a curve, is equal to the space that would be described by means of the sole action of that force, and moreover it is parallel to the direction of that force.* Wherefore, if the disturbing force be *constant*, it is easily inferred from LEMMAS X and IX, and indeed is shown in all books on Mechanics, that *the errors are as the squares of the times in which they are generated.* Also, if the disturbing forces be *nearly constant*, then *the errors are as the squares of the times quam proxime.* But these conclusions, the same as those which Note 118 of the Jesuits, Le Seur and Jacquier, (see Glasgow edit. 1822.) leads to, do not prove the assertion of Newton in the corollary under consideration, inasmuch as they are general for all curves, and apply not to *similar* curves in particular.

48. Now let a curve similar to the above be constructed; and completing the figure, let the points corresponding to A, B, &c. be denoted by A′, B′, &c. and let the times in which the similar parts of these curves, viz. B D, B′ D′; D E, D′ E′; E C, E′ C′ are described, be in the ratio t : t′. Then the times in which, by the *same disturbing force*, the spaces B F, B′ F′; F G, F′ G′; G b, G′ b′ are described, are in the ratio of t : t′. Hence, "in ipso motus initio" (by LEMMA X) we have

$$BF : B'F' : : t^2 : t'^2$$

$$FG : F'G' : : t^2 : t'^2$$

&c. &c.

and therefore,

$$BF + FG + \&c. : B'F' + F'G' + \&c. : : t^2 : t'^2$$

But, (by 15,)

$$B F + F G + \&c. = \text{the error } C c,$$

and

$$B' F' + F' G' + \&c. = \text{the error } C' c',$$

and the times in which B C, B′ C′ are described, are in the ratio t : t′. Hence then

$$C c : C' c' : : t^2 : t'^2$$

or *The ERRORS arising from equal forces, applied at corresponding points, disturbing the motions of bodies in similar curves, which describe similar parts of those curves in proportional times, are as the squares of the times in which they are generated* EXACTLY, *and not "quam proxime."*

Hence Newton appears to have neglected to investigate this corollary. The corollary indeed did not merit any great attention, being limited by several restrictions to very particular cases.

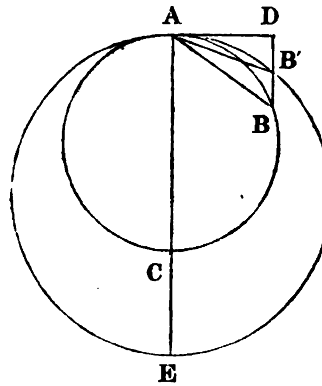
It would seem from this and the last No. that Newton's meaning in the forces being "similarly applied," is merely that they are to be applied at corresponding points, and do not *necessarily* act in directions *similarly* situated with respect to the curves.

For explanation with regard to the other corollaries, see 46.

49. LEMMA XI. "*Finite Curvature.*" Before we can form any precise notion as to the curvature at any point of a curve's being *Finite*, *Infinite* or *Infinitesimal*, some method of measuring curvature in general must be devised. This measure evidently depends on the *ultimate* angle contained by the chord and tangent (A B, A D) or on the *angle of contact.* Now, although this angle can have no finite value when singly considered, yet when two such angles are compared, their ratio may be finite, and if any known curvature be assumed of a standard magnitude, we shall have, by the equality between the ratios of the angles of contact and the curvatures, the curvature at any point in any curve whatever. In practice, however, it is more commodious to compare the subtenses of the angles of contact (which may be considered circular arcs, see LEMMA VII, having radii in a ratio of equality, and therefore are accurate measures of them), than the angles themselves.

50. Ex. 1. Let the circumference of a circle be divided into any number of equal parts and the points of division being joined, let there be a tangent drawn at every such point meeting a perpendicular let fall from the next point; then it may easily be shown that these perpendiculars or subtenses are all equal, and if the number of parts be increased, and their

magnitude diminished, *in infinitum*, they will have a ratio of equality. Hence, *the* CIRCLE *has the same curvature at every point, or it is a curve of uniform curvature.*



51. Ex. 2. Let two circles touch one another in the point A, having the common tangent A D. Also let B D be perpendicular to A D and cut the circle A D in B′. Join A B, A B′. Then since A B, A B′ are ultimately equal to A D (LEMMA VII) they are equal to one another, and consequently the limiting ratio of B D and B′ D, is that of the curvatures of the respective circles A C, A D (by 17.)

But, by the nature of the circle,

$$\mathrm{A\,D}^2 = 2\,\mathrm{R} \times \mathrm{D\,B'} - \mathrm{D\,B'}^2 = 2\,\mathrm{r} \times \mathrm{D\,B} - \mathrm{D\,B}^2$$

R and r being the radii of the circles.

Therefore

$$\mathrm{L} + 1 = \frac{\mathrm{D\,B}}{\mathrm{D\,B'}} = \frac{2\,\mathrm{R} - \mathrm{D\,B'}}{2\,\mathrm{r} - \mathrm{D\,B}}$$

and equating homogeneous terms we have

$$\mathrm{L} = \frac{\mathrm{R}}{\mathrm{r}},$$

i. e. *The curvatures of circles are inversely as their radii.*

52. Hence, if the curvature of the circle whose radius is 1, (inch, foot, or any other measure,) be denoted by C, that of any other circle whose radius is r, is

$$\frac{\mathrm{C}}{\mathrm{r}}.$$

53. Hence, if the radius r of a circle compared with 1, be *finite*, its curvature compared with C, is *finite*; if r be *infinite* the curvature is *infinitesimal*; if r be *infinitesimal* the curvature is *infinite*, and so on through all the higher orders of *infinites* and *infinitesimals*. By infinites and infinitesimals are understood quantities indefinitely great or small.

The above sufficiently explains why curvature, compared with a given standard (as C), can be said to be *finite* or *indefinite*. We are yet to show the reason of the restriction to curves of *finite curvature*, in the enunciation of the LEMMA.

54. The circles which pass through A, B, G; a, b, g, (fig. LEMMA XI)

have the same tangent A D with the curve and the same subtenses. Hence (49. and 52.) these circles *ultimately* have the same *curvature* as the curve, i. e. A I is the diameter of that circle which has the same curvature as the curve at A. Hence, according as A I is finite or indefinite, the curvature at A is so likewise, compared with that of circles of finite radius.

Now A G ultimately, or

$$\mathrm{A\,I} = \frac{\mathrm{A\,B}^2}{\mathrm{B\,D}}$$

whether A I be finite or not. If finite, B D $\propto$ A B^2, as we also learn from the text.

55. If the curvature be infinitesimal or A I infinite; then since $\frac{\mathrm{A\,B}^2}{\mathrm{B\,D}}$ is infinite, B D must be infinitely less than A B^2, or, A B being always considered in its ultimate state an infinitesimal of the first order, B D is that of the third order, i. e. B D $\propto$ A B^3. The converse is also true.

Ex. In the cubical parabola, the abscissa $\propto$ as the cube of the ordinate; hence at its vertex the curvature is infinitely small. At other points, however, of this curve, as we shall see hereafter, the curvature is finite.

To show at once the different proportions between the subtenses of the angles of contact and the conterminous arcs, corresponding to the different orders of infinitesimal or infinite curvatures, and to make intelligible this intricate subject, let A B ultimately considered be indefinitely small compared with 1; then since $\frac{\mathrm{A\,B}^2}{\mathrm{A\,B}} = \mathrm{A\,B}$, A B^2 is infinitesimal compared with A B; and generally $\frac{\mathrm{A\,B}^n}{\mathrm{A\,B}^{n-1}} = \mathrm{A\,B}$, shows that A B^n is infinitely small compared with A B^{n-1} so that *the different orders of infinitesimals may be correctly denoted by*

$$\mathrm{A\,B},\ \mathrm{A\,B}^2,\ \mathrm{A\,B}^3,\ \mathrm{A\,B}^4,\ \&c.$$

Also since 1 is infinite compared with the infinitesimal A B, and A B compared with A B^2, &c. *the different orders of infinites* may be represented by

$$\frac{1}{\mathrm{A\,B}},\ \frac{1}{\mathrm{A\,B}^2},\ \frac{1}{\mathrm{A\,B}^3},\ \frac{1}{\mathrm{A\,B}^4},\ \&c.$$

56. Hence if the curvature at any point of a curve be infinitesimal in the second degree

$$\frac{AB^2}{BD} \propto \frac{1}{AB^2}, \text{ and } BD \propto AB^4, \text{ and conversely.}$$

And generally, if the curvature be infinitesimal in the n^{th} degree,

$$\frac{AB^2}{BD} \propto \frac{1}{AB^n}, \text{ and } BD \propto AB^{n+2}, \text{ and conversely.}$$

Again, if the curvature be infinite in the n^{th} degree,

$$\frac{AB^2}{BD} \propto AB^n, \text{ and } BD \propto AB^{2-n}, \text{ and conversely.}$$

The parabolas of the different orders will afford examples to the above conclusions.

57. The above is sufficient to explain the first case of the LEMMA.

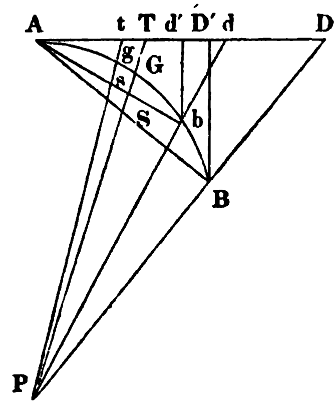
Case 2. presents no difficulty; for b d, B D being inclined at any equal angles to A D, they will be parallel and form, with the perpendiculars let fall from b, B upon A D, similar triangles, whose sides being proportional, the ratio between B D, b d will be the same as in Case 1.

Case 3. If B D converge, i. e. pass through when produced to a given point, b d will also, and ultimately when d and D move up to A, the difference between the angles A d b, A D B will be less than any that can be assigned, i. e. B D and b d will be ultimately parallel; which reduces this case to Case 2. (See Note 125. of PP. Le Seur and Jacquier.)

Instead of passing through a given point, B D, b d may be supposed to touch perpetually any given curve, as a circle for instance, and B D will still $\propto$ A D^2; for the angles D, d are ultimately equal, inasmuch as from the same point A there can evidently be but one line drawn touching the circle or curve.

Many other laws determining B D might be devised, but the above will be sufficient to illustrate Newton's expression, "or let B D be determined by any other law whatever." It may, however, be farther observed that this law must be definite or such as will *fix* B D. For instance, the LEMMA would not be true if this law were that B D should *cut* instead of *touch* the given circle.

58. LEMMA XI. COR. II. It may be thus explained. Let P be the given point towards which the sagittæ S G, s g, bisecting the chords A B, A b, converge. S G, s g shall ultimately be as the squares of A B, A b, &c.



For join P B, P b and produce them, as also P G, P g, to meet the tangent in D, d, T, t. Then if B and b move up to A, the angles T P D, t P d, or the differences between the angles A T P and A D P, and between A t P and A d P, may be diminished without limit; that is, (LEMMA I), the angles at T, D and at t, d are ultimately equal. Hence the triangles A T S, A D B are similar, as likewise are A t s, A d b.

Consequently

$$ST : DB :: AS : AB :: 1 : 2$$

and

$$st : db :: As : Ab :: 1 : 2$$

and

$$\therefore ST : st :: DB : db$$

Also by LEMMA VII,

$$ST : st :: SG : sg$$

and by LEMMA XI, Case 3,

$$DB : db :: AB^2 : Ab^2$$

$$\therefore SG : sg :: AB^2 : Ab^2$$

Q. e. d.

Moreover, it hence appears, that *the sagittæ which cut the chords, in ANY GIVEN RATIO WHATEVER, and tend to a given point, have ultimately the same ratio as the subtenses of the angles of contact, and are as the squares of the corresponding arcs, chords, or tangents.*

59. LEMMA XI. COR. III. If the velocity of a body be constant or "given," the space described is proportional to the time t. Hence $AB \propto t$, and $\therefore SG \propto AB^2 \propto t^2$.

60. LEMMA XI. COR. IV. Supposing B D, b d at right angles to A D (and they have the same proportion when inclined at a given angle to A D, and also when tending to a given point, &c.) we have

$$\triangle ADB : \triangle Adb :: \frac{AD \times DB}{2} : \frac{Ad \times db}{2}$$

$$:: \frac{DB}{db} \times AD : Ad$$

$$:: \frac{AD^2}{Ad^2} \times AD : Ad$$

$$:: AD^3 : Ad^3.$$

Also

$$\triangle ADB : \triangle Adb :: \frac{AD}{Ad} \times DB : db$$

$$:: \frac{\sqrt{DB}}{\sqrt{db}} \times DB : db$$

$$:: (DB)^{\frac{3}{2}} : (db)^{\frac{3}{2}}.$$

It may be observed here, that the tyro, on reverting to LEMMA IX, usually infers from it that

$\triangle ADB \propto AD^2$ and does not $\propto AD^3$,

but then he does not consider that A D, in LEMMA IX, cuts or makes a *finite* angle with the curve, whereas in LEMMA XI it touches the curve.

61. LEMMA XI. COR. V. Since in the common parabola the abscissa $\propto$ square of the ordinate, and likewise B D or A C $\propto$ AD^2 or CD^2, it is evident that the curve may ultimately be considered a parabola.

This being admitted, we learn from Ex. 1, No. 4, that the curvilinear area A C B = $\frac{2}{3}$ of the rectangle C D. Whence the curvilinear area A B D = $\frac{1}{3}$ of C D = $\frac{2}{3}$ of the triangle A B D, or the area A B D $\propto$ triangle A B D $\propto$ AD^3, &c. (by Cor. 4.) So far B D, b d have been considered at right angles to A D. Let them now be inclined to it at a given angle, or let them tend to a given point, or "be determined by any other law;" then (LEMMA, Case 3, and No. 25) B D, b d will ultimately be parallel. Hence, B D′, b d′ (fig. No. 26) being the corresponding subtenses perpendicular to A D, it is plain enough that the ultimate differences between the curvilinear areas A B D, A B D′ and between A b d, A b d′ are the similar triangles B D D′, b d d′, which differences are therefore as BD^2, bd^2, or as AB^4, Ab^4, i. e. $BDD' \propto AB^4$.

But we have shown that A B D $\propto$ AB^3.

Consequently

$$ABD' = ABD \mp BDD' = a \times AB^3 \mp b \times AB^4 = AB^3(a \mp b \times AB)$$

and $b \times AB$ being indefinite compared with a, (see Art. 6,)

$$ABD' = a \times AB^3 \propto AB^3.$$

Q. e. d.

SCHOLIUM TO SECTION I.

62. What Newton asserts in the Scholium, and his commentators Le Seur and Jacquier endeavour (*unsuccessfully*) to elucidate, with regard to the different orders of the angles of contact or curvatures, may be briefly explained, thus.

Let $DB \propto AD^m$. Then the diameter of curvature, which equals $\frac{AD^2}{DB}$ (see No. 22 and 24), $\propto AD^{2-m}$. Similarly if $DB \propto AD^n$, the diameter of curvature $\propto AD^{2-n}$. Hence D and D′ represents these diameters, we have

$$\frac{D}{D'} = \frac{a \times AD^{2-m}}{a' \times AD^{2-n}} = \frac{a}{a'} AD^{n-m} \text{ (a and a′ being finite)}$$

and if $n = 2$ or D′ be *finite*, then D will be *finite, infinitesimal*, or *infinite*, according as $m = 2$, or is any number, (whole, fractional, or even transcendental) less than 2, or any number greater than 2. Again, if $m = n$ then D compared with D′ is finite, since $D : D' :: a : a'$. If m be less than n in any finite degree, then $n - m$ is positive, and D is always infinitely less than D′. If m be greater than n, then

$$\frac{D}{D'} = \frac{a}{a'} \times \frac{1}{AD^{m-n}}$$

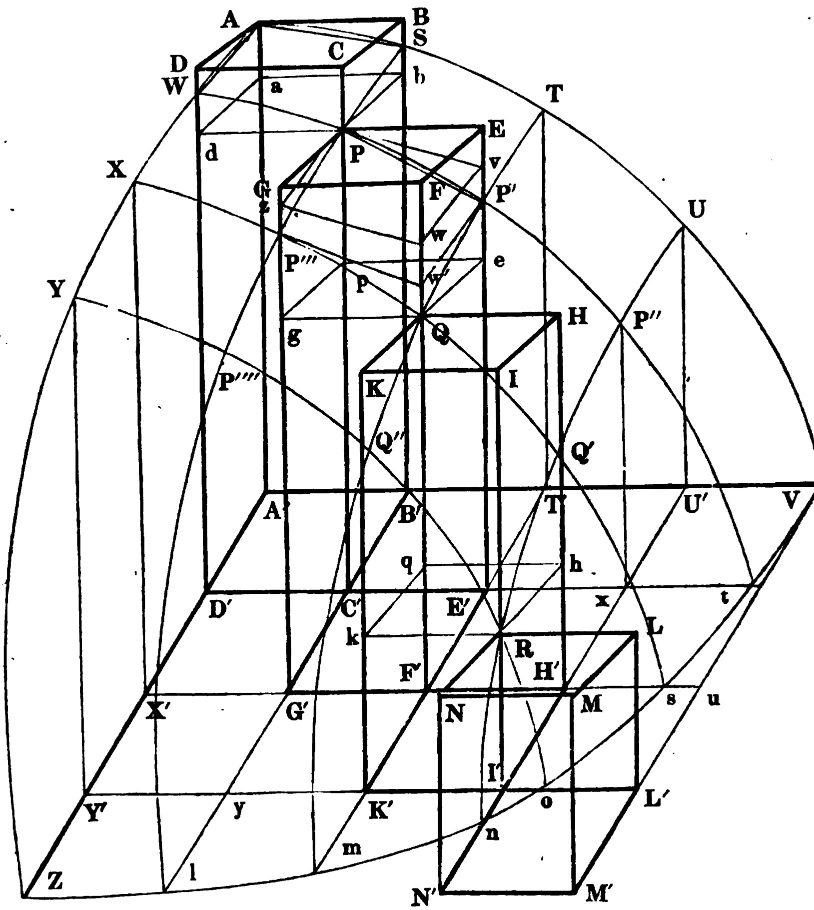
and $m - n$ being positive, D is always infinite compared with D′.

Hence then, there is no limit to the orders of diameters of curvature, with regard to infinite and infinitesimal, and consequently not to the curvatures.

63. In this Scholium Newton says, that "Those things which have been demonstrated of curve lines and the surfaces which they comprehend are easily applied to the curve surfaces and contents of solids." Let us attempt this application, or rather to show,

1st, *That if any number of parallelopipeds of equal bases be inscribed in any solid, and the same number having the same bases be also circumscribed*

about it; then the number of these parallelopipeds being increased and their magnitude diminished IN INFINITUM, the ultimate ratios which the aggregates of the inscribed and circumscribed parallelopipeds have to one another and to the solid, are ratios of equality.



Let A S T U V Z Y X W A be any portion of a solid cut off by three planes A A′ V, A A′ Z and Z A′ V, passing through the same point A′, and perpendicular to one another. Also let the intersections of these planes with one another be A A′, A′ V, A′ Z, and with the surface of the solid be A U V, A Y Z and Z l V. Moreover let A′ V, A′ Z be each divided into any number of equal parts in the points B′, T′, U′; D′, X′, Y′, and through them let planes, parallel to A A′ Z and A A′ V respectively, be supposed to pass, whose intersections with the planes A A′ V, A A′ Z

shall be S B′, T T′, U U′; W D′, X X′, Y Y′, and with the plane A′ Z V, l B′, m T′, n U′; t D′, s X′, o Y′, respectively. Again, let the intersections of these planes with the curve surface be S P l, T Q m, U R n; W P t, X Q s, Y R o respectively. Also suppose their several mutual intersections to be P C′, P′ E′, P″ x, P‴ G′, Q F′, Q′ H′, Q″ K′, &c.; those of these planes taken in pairs and of the plane A′ Z V, being the points C′, E′, x, G′, F′, H′, K′, I′, &c. and those of these pairs of planes and of the curve surface, the points P, P′, P″, P‴, Q, Q′, Q″, R, &c.

Now the planes, passing through B′, T′, U′, being all parallel to A A′ Z, are parallel to one another and perpendicular to A A′ V. Also because the planes passing through D′, X′, Y′ are parallel to A A′ V, they are parallel to one another, and perpendicular to A A′ Z. Hence (Euc. B. XI.) S B′, T T′, U U′, W D′, X X′, Y Y′, as also P C′, P′ E′, P″ x, P‴ G′, Q F′, Q′ H′, Q″ K′, &c. &c. are parallel to A A′ and to one another. It is also evident, for the same reasons, that B′ l, T′ m, U′ n, are parallel to A′ Z and to one another, as also are D′ t, X′ s, Y′ o to A′ V and to one another. Hence also it follows that A′ B′ C′ D′, B′ C′ E′ T′, &c. are rectangles, which rectangles, having their sides equal, are themselves equal.

Again, from the points A, P, Q, R in the curve surface, draw A B, A D; P E, P G; Q H, Q K; R L, R N parallel to A′ B′, A′ D′; C′ E′, C′ G′; F′ H′, T′ K′, I′ o, I′ n and meeting B′ S, D′ W; E′ P′, G′ P‴; H′ Q′, K′ Q″ produced in the points B, D; E, G; H, K, respectively. Then complete the rectangles A C, P F, Q I which, being equal and parallel to A′ C′, C′ F′, F′ I′, will evidently, when C′ P, F′ Q, I′ R are produced to C, F, I, complete the rectangular parallelopipeds A C′, P F′, Q I′. Moreover, supposing F′ I′ the last rectangle wholly within the curve Z V produce K′ I′, H′ I′ and make I′ L′, I′ N′ equal K′ I′, H′ I′, and complete the rectangle I′ M′. Also complete the parallelopiped R M′.

Again, produce E P, G P, H Q, K Q; L R, N R to the points d, b; g, e; k, h, and complete the rectangles P a, Q p, R q thereby dividing the parallelopipeds A C′, P F′, Q I′, each into two others, viz. A P, a C′; P Q, p F′; Q R, q I′.

Now the difference between the sum of the inscribed parallelopipeds a C′, p F′, q I′, and that of the circumscribed ones A C′, P F′, Q I′, R M′, is evidently the sum of the parallelopipeds A P, P Q, Q R, R M′; that is, since their bases are equal and the altitudes I′ R′, R I, Q F, P C are together equal to A A′, this difference is equal to the parallelopiped A C′. In the same manner if a series of inscribed and circumscribed

rectangular parallelopipeds, having the bases B′ E′, E′ H′, H′ L′, be constructed, the difference between their aggregates will equal the parallelopiped whose base is B′ E′ and altitude S B′, and so on with every series that can be constructed on bases succeeding each other *diagonally*. Hence then the difference between the sums of all the parallelopipeds that can be inscribed in the curve surface A Z V and circumscribed about it, is the sum of the parallelopipeds whose bases are each equal to A′ C′ and altitudes are A A′, S B′, T T′, U U′, W D′, X X′, Y Y′. Let now the number of the parts A′ B′, B′ T′, T′ U′, U′ V, and of the parts A D′, D′ X′, X′ Y′, Y′ Z be increased, and their magnitude diminished *in infinitum*, and it is evident the aforesaid sum of the parallelopipeds, which are comprised between the planes A A′ Z, S B′ l and between the planes A A′ V, W D′ t, will also be diminished without limit; that is, the difference between the inscribed and circumscribed whole solid is ultimately less than any that can be assigned, and these solids are *ultimately* equal, and a fortiori is the intermediate curve-surfaced solid equal to either of them (see LEMMA I and Art. 6.) Q. e. d.

Hitherto only such *portions* of solids as are bounded by three planes perpendicular to one another, and passing through the same point, have been considered. But since a *complete curve-surfaced* solid will consist of four such portions, it is evident that what has been demonstrated of any one portion must hold with regard to the whole. Moreover, if the solid should not be curve-surfaced throughout, but have one, two, or three plane faces, there will be no difficulty in modifying the above to suit any particular case.

2dly, *If in two curve-surfaced solids there be inscribed two series of parallelopipeds, each of the same number; and ultimately these parallelopipeds have to each other a given ratio, the solids themselves have to one another that same ratio.*

This follows at once from the above and the composition of ratios.

3dly, *All the corresponding edges or sides, rectilinear or curvilinear, of similar solids are proportionals; also the corresponding surfaces, plane or curved, are in the duplicate ratio of the sides; and the volumes or contents are in the triplicate ratio of the sides.*

When the solids have plane surfaces only, the above is shown to be true by Euclid.

When, however, the solids are curve-surfaced, wholly or in part, we must define them *to be similar when any plane-surfaced solid whatever being inscribed in any one of them, similar ones may also be inscribed in the*

others. Hence it is evident that the corresponding plane surfaces are similar, and consequently, by Lemma V, the corresponding edges are proportional, and the corresponding plane surfaces are in the duplicate ratio of these edges or sides. Moreover, if the same number of similar parallelopipeds be inscribed in the solids, and that number be indefinitely increased, it follows from 63. 1 and the composition of ratios, that the curved surfaces are proportional to the corresponding plane surfaces, and therefore in the duplicate ratio of the corresponding edges; and also that the contents are proportional to the corresponding inscribed parallelopipeds, or (by Euclid) in the triplicate ratio of the edges.

These three cases will enable the student of himself to pursue the analogy as far as he may wish. We shall "leave him to his own devices," after cautioning him against supposing that a curved-surface, at any point of it, has a certain fixed degree of curvature or deflection from the tangent-plane, and therefore that there is a sphere, touching the tangent-plane at that point, whose diameter shall be the limit of the diameters of all the spheres that can be made to touch the tangent-plane or curved-surface —analogously to A I in Lemma XI. Every curvilinear section of a curved-surface, made by a plane passing through a given point, has at that point a different curvature, the curved-surface being taken in the general sense; and it is a problem of Maxima and Minima *To determine those sections which present the greatest and least degrees of curvature.*

The other points of this Scholium require no particular remarks. If the student be desirous of knowing in what consists the distinction between the obsolete methods of Exhaustions, Indivisibles, &c. and that of prime and ultimate ratios, let him go to the original sources—to the works of Archimedes, Cavalerius, &c.

64. Before we close our comments upon this very important part of the *Principia*, we may be excused, perhaps, if we enter into the detail of the Principle delivered in Art. 6, which has already afforded us so much illustration of the text, and, as we shall see hereafter, so many valuable results. We have thence obtained a number of the ordinary rules for deducing indefinite forms from given definite functions of one variable; and it will be confessed, by competent and candid judges, that these applications of the principle strongly confirm it. Enough has indeed been already developed of the principle, to prove it clearly divested of all the metaphysical obscurities and inconsistencies, which render the methods of Fluxions, Differential Calculus, &c. &c. so objectionable as to their logic, and which have given rise to so many theories, all tending to establish

the same rules. It is incredible that the great men, who successively introduced their several theories, should have been satisfied with the reasonings by which they attempted to establish them. So many conflicting opinions, as to the principles of the science, go only to show that all were founded in error. Although it is generally difficult, and often impossible, for even the most sharp-sighted of men, to discern truth through the clouds of error in which she is usually enveloped, yet, when she does break through, it is with such distinct beauty and simplicity that she is instantly recognized by all. In the murkiness around her there are indeed false lights innumerable, and each passing meteor is in turn, by many observers, mistaken for the real presence; but these instantly vanish when exposed to the refulgent brightness of truth herself. Thus we have seen the various systems of the world, as devised by Ptolemy, Tycho Brahe, and Descartes, give way, by the unanimous consent of philosophers, to the demonstrative one of Newton. It is true, the principle of gravitation was received at first with caution, from its non-accordance with astronomical observations; but the moment the cause of this discrepancy, viz. the erroneous admeasurement of an arc of the meridian, was removed, it was hailed universally as truth, and will doubtless be coeval with time itself. The Theories relative to quantities indefinitely variable, present an argument from which may be drawn conclusions directly opposite to the above. Newton himself, dissatisfied with his Fluxions, produces PRIME AND ULTIMATE RATIOS, and again, dissatisfied with these, he introduces the idea of Moments in the second volume of the Principia. He is every where constrained to apologize for his obscurities, first in his Fluxions for the use of time and velocities, and then again in the Scholium, at the end of Sect. I of the Principia, (and in this instance we have shown how little it avails him) for reasoning upon *nothings*. After Newton comes Leibnitz, his great though dishonest rival, (we may so designate him, such being evidently the sentiments of Newton himself), who, bent upon obliterating all traces of his spoil, melts it down into another form, but yet falls into greater errors, as to the true nature of the thing, than the discoverer himself. From his Infinitesimals, considered as absolute *nothings* of the different orders, nothing can be logically deduced, unless by *Him* (we speak with reverence) who made all things from *nothing*. Such *fiats* we mortals cannot issue with the same effect, nor do we therefore admit in science, finite and tangible consequences deduced from the arithmetic of absolute nothings, be they ever so many. Then we have a number of theories promulgated by D'Alembert, Euler, Simpson, Marquise L'Hopital, &c. &c.

all more or less modifications of the others—all struggling to establish and illustrate what the great inventor, with all his almost supernatural genius, failed to accomplish. All these diversities in the views of philosophers make, as it has been already observed, a strong argument that truth had not then unveiled herself to any of them. Newton strove most of any to have a full view, but he caught only a glimpse, as we may perceive by his remaining dissatisfied with it. Hence then it appears, to us at least, that the true metaphysics of the doctrine of quantities indefinitely variable, remain to this day undiscovered. But it may be asked, after this sweeping conclusion, how comes it that the results and rules thence obtained all agree in form, and in their application to physics produce consequences exactly in conformity with experience and observation? The answer is easy. These forms and results are accurately true, although illogically deduced, from a mere *compensation of errors*. This has been clearly shown in the general expression for the subtangent (Art. 29), and all the methods, not even Lagrange's Calcul des Fonctions excepted, are liable to the paralogism. Innumerable other instances might be adduced, but this one we deem amply sufficient to warrant the above assertion.

After these preliminary observations upon the state of darkness and error, which prevails to this day over the scientific horizon, it may perhaps be expected of us to shine forth to dispel the fog. But we arrogate to ourselves no such extraordinary powers. All we pretend to is *self-satisfaction* as to the removal of the difficulties of the science. Having engaged to write a Commentary upon the Principia, we naturally sought to be satisfied as to the correctness of the method of Prime and Ultimate Ratios. The more we endeavoured to remove objections, the more they continually presented themselves; so that after spending many months in the fruitless attempt, we had nearly abandoned the work altogether; when suddenly, in examining the method of *Indeterminate Coefficients* in Dr. Wood's Algebra, it occurred that the aggregates of the coefficients of the like powers of the indefinite variable, must be separately equal to zero, not because the variable might be assumed equal to zero, (which it never is, although it is capable of indefinite diminution,) but because of the different powers being essentially different from, and forming no part of one another.

From this a train of reflections followed, relative to the treatment of homogeneous *definite* quantities in other branches of Algebra. It was soon perceptible that any equation put $= 0$, consisting of an aggregate of

different quantities incapable of amalgamation by the opposition of *plus* and *minus*, must give each of these quantities equal to zero. Reverting to indefinites, it then appeared that their whole theory might be developed on the same principles, and making trial as in Art. 6, and the subsequent parts of the preceding commentary, we have satisfied ourselves most fully of having thus hit upon a method of clearing up all the difficulties of what we shall henceforth, contrary to the intention expressed in Art. 7, entitle

THE CALCULUS

OF

INDEFINITE DIFFERENCES.

65. A *constant quantity* is such, that from its very nature it cannot be made less or greater.

Constants, as such quantities may briefly be called, are denoted generally by the first letters of the alphabet,

$$a, b, c, d, \&c.$$

A definite quantity is a GIVEN VALUE *of a quantity essentially variable.*

Definite quantities are denoted by the last letters of the alphabet, as

$$z, y, x, w, \&c.$$

An INDEFINITE *quantity is a quantity essentially variable through all degrees of diminution or of augmentation short of absolute* NOTHINGNESS *or* INFINITUDE.

Thus the ordinate of a curve, considered generally, is an *indefinite*, being capable of every degree of diminution. But if any particular value, as that which to a given abscissa, for instance, be fixed upon, this value is *definite*. All abstract numbers, as 1, 2, 3, &c. and quantities absolutely fixed, are *constants*.

66. The difference between two definite values of the same quantity (y) is a *definite* quantity, and may be represented by

$$\Delta y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

adopting the notation of the Calculus of Finite (or *definite*) Differences.

In the same manner the difference between two definite values of Δ y is a definite quantity, and is denoted by

$$\Delta (\Delta y)$$

or more simply by

$$\Delta^2 y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and so on to

$$\Delta^n y.$$

67. The difference between a *Definite* value and the *Indefinite* value of any quantity y is *Indefinite*, and we call it the *Indefinite Difference* of y, and denote it, agreeably to the received algorithm, by

$$d y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

In the same manner

$$d (d y)$$

or

$$d^2 y$$

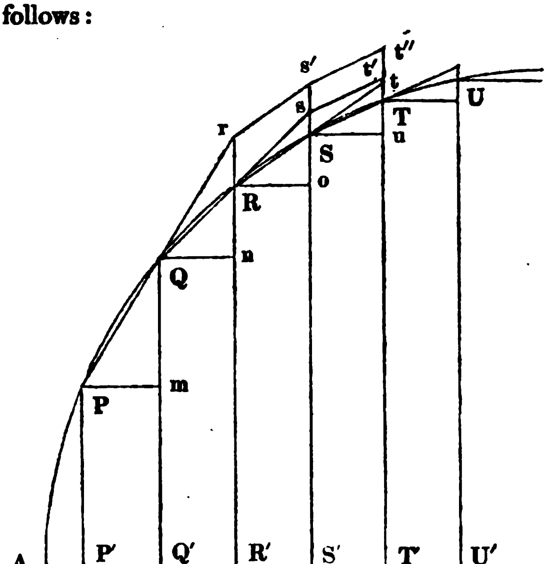
the *Indefinite Difference* of the *Indefinite Difference* of y, or the second indefinite difference of y.

Proceeding thus we arrive at

$$d^n y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

which means the n^{th} indefinite difference of y.

68. *Definite* and *Indefinite* Differences admit of being represented by lines, as follows:



Let P P' = y be any fixed or definite ordinate of the curve A U, and taking P' Q' = Q' R' = R' S' = &c. let ordinates be erected meeting the curve in Q, R, S, T, &c. Join P Q, Q R, R S, &c. and produce them to meet the ordinates produced in r, s, t, &c. Also draw r s', s t',

&c. parallel to R S, S T, &c. and draw s t″, &c. parallel to s t′, &c.; and finally draw P m, Q n, R o, &c. perpendicular to the ordinates.

Now supposing not only P P′ but also Q Q′, R R′, &c. fixed or definite; then

$$\begin{aligned} Q\,m &= Q\,Q' - P\,P' = \Delta\,P\,P' = \Delta\,y \\ R\,r &= n\,r - n\,R = Q\,m - R\,n = \Delta\,Q\,m \\ &= \Delta\,(\Delta\,P\,P') = \Delta^2\,P\,P' = \Delta^2\,y \\ s\,s' &= S\,s - S\,s' = S\,s - R\,r = \Delta\,R\,r \\ &= \Delta^3\,y \\ t\,t'' &= t\,t' - t'\,t'' = t\,t' - s\,s' = \Delta\,s\,s' \\ &= \Delta\,(\Delta^3\,y) = \Delta^4\,y \end{aligned}$$

and so on to any extent.

But if the equal parts P′ Q′, Q′ R′, &c. be arbitrary or indefinite, then Q m, R r, s s′, t t″, &c. become so, and they represent the several *Indefinite Differences* of y, viz.

$$d\,y,\ d^2\,y,\ d^3\,y,\ d^4\,y,\ \&c.$$

69. The reader will henceforth know the distinction between *Definite* and *Indefinite Differences.* We now proceed to establish, of *Indefinite Differences,* the

FUNDAMENTAL PRINCIPLE.

It is evidently a truth perfectly axiomatic, that *No aggregate of* INDEFINITE *quantities can be a definite quantity, or aggregate of definite quantities, unless these aggregates are equal to zero.*

It may be said that $(a - x) + (a + x) = 2\,a$, in which (x) is indefinite, and (a) constant or definite, is an instance to the contrary; but then the reply is, $a - x$ and $a + x$ are not *indefinites* in the sense of Art. 65.

70. *Hence if in any equation*

$$A + B\,x + C\,x^2 + D\,x^3 + \&c. = 0$$

A, B, C, &c. be *definite quantities* and x an *indefinite quantity;* then we have

$$A = 0,\ B = 0,\ C = 0,\ \&c.$$

For $B\,x + C\,x^2 + D\,x^3 + \&c.$ cannot equal $-$ A unless $A = 0$. But by transposing A to the other side of the equation, it does $= -$ A. Therefore $A = 0$ and consequently

$$B\,x + C\,x^2 + D\,x^3 + \&c. = 0$$

or

$$x\,(B + C\,x + D\,x^2 + \&c.) = 0$$

But x being *indefinite* cannot be equal to 0; $\therefore$

$$B + C x + D x^2 + \&c. = 0$$

Hence, as before, it may be shown that $B = 0$, and therefore

$$x (C + D x + \&c.) = 0$$

Hence $C = 0$, and so on throughout.

71. Again, *if in the equation*

$$A + B x + B' y + C x^2 + C' x y + C'' y^2 + D x^3 + D' x^2 y + D'' x y^2 + D''' y^3 + \&c.$$

A, B, B′, C, C′, C″, D, &c. *be definite quantities*, and x, y *INDEFINITES*; *then*

$$\left.\begin{aligned} A &= 0 \\ B x + B' y &= 0 \\ C x^2 + C' x y + C'' y^2 &= 0 \end{aligned}\right\} \text{when } y \text{ is a function of } x.$$
$$\&c. = 0$$

For, let $y = z x$, then substituting

$$A + x (B + B' z) + x^2 (C + C' z + C'' z^2)$$
$$+ x^3 (D + D' z + D'' z^2 + D''' z^3) + \&c. = 0$$

Hence by 70,

$$A = 0, \; B + B' z = 0, \; C + C' z + C'' z^2 = 0, \; \&c.$$

and substituting $\frac{y}{x}$ for z and reducing we get

$$A = 0, \; B x + B' y = 0, \; \&c.$$

In the same manner, if we have an equation involving three or more indefinites, it may be shown that the aggregates of the homogeneous terms must each equal zero.

This general principle, which is that of *Indeterminate Coefficients* legitimately established and generalized, (the ordinary proofs divide $B x + C x^2 + \&c. = 0$ by x, which gives $B + C x + D x^2 + \&c. = \frac{0}{x}$ and not 0; x is then put $= 0$, and thence *truly* results $B = \frac{0}{0}$, which instead of being 0, may be any quantity whatever, as we know from algebra; whereas in 70, by considering the *nature* of x, and the absurdity of making it $= 0$ we avoid the paralogism) conducts us by a near route to the *Indefinite Differences of functions of one or MORE variables.*

72. *To find the Indefinite Difference of any function of* x.

Let $u = f x$ denote the function.

Then d u and d x being the indefinite differences of the function and of x itself, we have

$$u + d u = f (x + d x)$$

Assume

$$f (x + d x) = A + B \, d x + C \, d x^2 + \&c.$$

A, B, &c. being independent of dx or definite quantities involving x and constants; then

$$u + du = A + B\,dx + C\,dx^2 + \&c.$$

and by 71, we have

$$u = A,\ du = B.dx$$

Hence then this general rule,

The INDEFINITE DIFFERENCE *of any function of* x, fx, *is the second term in the developement of* $f(x + dx)$ *according to the increasing powers of* dx.

Ex. Let $u = x^n$. Then it may easily be shown independently of the Binomial Theorem that

$$(x + dx)^n = x^n + n.x^{n-1}dx + P\,dx^2$$

$$\therefore d(x^n) = n.x^{n-1}dx$$

The student may deduce the results also of Art. 9, 10, &c. from this general rule.

73. *To find the indefinite difference of the product of two variables.*

Let $u = xy$. Then

$$u + du = (x + dx).(y + dy) = xy + x\,dy + y\,dx + dx\,dy$$

$$\therefore du = x\,dy + y\,dx + dx\,dy$$

and by 71, or directly from the homogeneity of the quantities, we have

$$du = x\,dy + y\,dx \quad \dots\dots\dots \quad (a)$$

Hence

$$d(xyz) = x\,d(yz) + yz\,dx$$

$$= xz\,dy + xy\,dz + yz\,dx \quad \dots \quad (b)$$

and so on for any number of variables.

Again, *required* $d.\frac{x}{y}$.

Let $\frac{x}{y} = u$. Then

$$x = yu, \text{ and } dx = u\,dy + y\,du$$

$$\therefore d\frac{x}{y} = du = \frac{dx}{y} - \frac{u}{y}dy$$

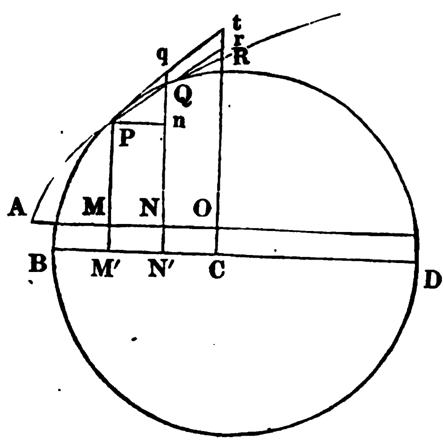
$$= \frac{y\,dx - x\,dy}{y^2} \quad \dots\dots\dots \quad (c)$$

Hence, and from rules already delivered, may be found the *Indefinite Differences* of any functions whatever of two or more variables. We refer the student to Peacock's Examples of the Differential Calculus for practice.

The result (a) may be deduced geometrically from the fig. in Art. 21. The sum of the indefinite rectangles A b, b A′ makes the Indefinite Difference.

We might, in this place, investigate the second, third, &c. Indefinite Differences, and give rules for the *maxima* and *minima* of functions of two or more variables, and extend the Theorems of Maclaurin and Taylor to such cases. Much might also be said upon various other applications, but the complete discussion of the science we reserve for an express Treatise on the subject. We shall hasten to deduce such results as we shall obviously want in the course of our subsequent remarks; beginning with the research of a general expression for the *radius of curvature* of a given curve, or for the radius of that circle whose deflection from the tangent is the same as that of the curve at the point of contact.

74. *Required the radius of curvature for any point of a given curve.*



Let A P Q R be the given curve, referred to the axis A O by the ordinate and abscissa P M, A M or y and x. P M being fixed let Q N, O R be any other ordinates taken at equal indefinite intervals M N, N O. Join P Q and produce it to meet O R in r; and let P t be the tangent at P drawn by Art. 29, meeting Q N, O R in q and t respectively. Again draw a circle (as in construction of LEMMA XI, or otherwise) passing through P and Q and touching the tangent P t, and therefore touching the curve; and let B D be its diameter parallel to A O.

Now

$$Q\,n = d\,y,\ P\,n = d\,x,\ P\,q = P\,Q\ (\text{LEMMA VII}) = \sqrt{(d\,x^2 + d\,y^2)} \text{ or } d\,s, \text{ if } s = \text{arc } A\,P.$$

Moreover let

$$P\,M' = y';$$

then it readily appears (see Art. 27) that $d\,s = \frac{R\,d\,x}{y}$, R being the radius of the circle.

Again

$$\begin{aligned} P\,q^2 &= Q\,q \times (Q\,q + 2\,Q\,N') \\ &= Q\,q\,(Q\,q + 2\,d\,y + 2\,y') \end{aligned}$$

or

$$(d\,s)^2 = Q\,q\left(Q\,q + 2\,d\,y + \frac{2\,R\,d\,x}{d\,s}\right)$$

But since

$$R\,t : Q\,q :: P\,r^2 : P\,Q^2 :: 4 : 1 \text{ (LEMMA XI)}$$

and

$$Q\,q : t\,r :: 1 : 2$$

$$\therefore R\,t = 2\,t\,r, \text{ or } R\,r = t\,r = 2\,Q\,q$$

$$\therefore Q\,q = \frac{t\,r}{2} = \frac{d^2 y}{2} \text{ (by Art. 68.)}$$

Consequently

$$d\,s^2 = \frac{d^2 y}{2}\left(\frac{d^2 y}{2} + 2\,d\,y + \frac{2\,R\,d\,x}{d\,s}\right)$$

$$= \frac{(d^2 y)^2}{4} + d\,y\;d^2 y + \frac{R\;d\,x\;d^2 y}{d\,s}$$

and equating *Homogeneous Indefinites*

$$d\,s^2 = \frac{R\;d\,x\;d^2 y}{d\,s}$$

$$\therefore R = \frac{d\,s^3}{d\,x\;d^2 y} = \frac{(d\,x^2 + d\,y^2)^{\frac{3}{2}}}{d\,x\;d^2 y}$$

$$= \frac{\left(1 + \frac{d\,y^2}{d\,x^2}\right)^{\frac{3}{2}}}{\frac{d^2 y}{d\,x^2}} \quad \ldots\ldots\ldots\ldots \quad (d)$$

the general expression for the radius of curvature.

Ex. 1. In the parabola $y^2 = a\,x$.

$$\therefore \frac{d\,y}{d\,x} = \frac{a}{2\,y}$$

and since when the curve is concave to the axis $d^2 y$ is negative,

$$-\frac{d^2 y}{d\,x^2} = -\frac{a}{2\,y^2}\cdot\frac{d\,y}{d\,x} = -\frac{a^2}{4\,y^3} =$$

$$\therefore R = \left(1 + \frac{a^2}{4\,y^2}\right)^{\frac{3}{2}} \times \frac{4\,y^3}{a^2}$$

$$= (4\,y^2 + a^2)^{\frac{3}{2}} \times \frac{1}{2\,a^2}$$

Hence at the vertex $R = \frac{a}{2}$, and at the extremity of the latus rectum,

$$R = \frac{2^{\frac{3}{2}}}{2}\,a = a\,\sqrt{2}.$$

Ex. 2. If p be the parameter or the double ordinate passing through the focus and 2 a the axis-major of any conic section, its equation is

$$y^2 = p\,x \pm \frac{p}{2\,a}x^2$$

Hence

$$2\,y\,d\,y = p\,d\,x \pm \frac{p}{a}\,x\,d\,x$$

and

$$2\,d\,y^2 + 2\,y\,d^2\,y = \pm\frac{p}{a}\,d\,x^2$$

$$\therefore \frac{d\,y}{d\,x} = \frac{p\left(1 \pm \frac{x}{a}\right)}{2\,y}$$

and

$$-\frac{d^2\,y}{d\,x^2} = \frac{p^2\left(1 \pm \frac{x}{a}\right)^2 \mp \frac{2\,p}{a}\,y^2}{4\,y^3}$$

$$\therefore R = \frac{\left\{4\,y^2 + p^2\left(1 \pm \frac{x}{a}\right)^2\right\}^{\frac{3}{2}}}{2\left\{p^2\left(1 \pm \frac{x}{a}\right)^2 \mp \frac{2\,p}{a}\,y^2\right\}}$$

which reduces to

$$R = \frac{\left\{p^2 + \frac{2\,p}{a}(2\,a \pm p)\,x + \frac{p}{a^2}(p \pm 2\,a)\,x^2\right\}^{\frac{3}{2}}}{2\,p^2}$$

Ex. 3. In the cycloid it is easy to show that

$$\frac{d\,y}{d\,x} = \sqrt{\frac{2\,r - y}{y}}$$

r being the radius of the generating circle, and x, y referred to the base or path of the circle.

$$\therefore \frac{d^2\,y}{d\,x^2} = -\frac{r}{y^2}$$

$$\therefore R = 2\,\sqrt{2\,r\,y} = 2 \text{ the normal.}$$

Hence it is an easy problem *to find the equation to the locus of the centres of curvature for the several points of a given curve.*

If y and x be the coordinates of the given curve, and Y and X those of the required locus, all referred to the same origin and axis, then the student will easily prove that

$$X = x - \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{d^2y}{dx^2}\right)}\left(1 + \frac{dy^2}{dx^2}\right)$$

and

$$Y = y + \frac{1 + \frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}}$$

which will give the equation required, by substituting by means of the equation to the given curve.

In the cycloid for instance

$$X = x + \sqrt{(2ry - y^2)}$$
$$Y = -y$$

whence it easily appears that the locus required is the same cycloid, only differing in position from the given one.

75. *Required to express the radius of curvature in terms of the polar co-ordinates of a curve, viz. in terms of the radius vector ϱ and traced-angle θ.*

$$\left.\begin{array}{l} x = \varrho \cos. \theta \\ \text{and} \\ y = \varrho \sin. \theta \end{array}\right\}$$

$\therefore$ taking the indefinite differences, and substituting in equation (d) of Art. 74, we get

$$R = \frac{\left(\varrho^2 + \frac{d\varrho^2}{d\theta^2}\right)^{\frac{3}{2}}}{2\frac{d\varrho^2}{d\theta^2} - \varrho\frac{d^2\varrho}{d\theta^2} + \varrho^2}$$

which by means of the equation to the curve will give the radius of curvature required.

Ex. 1. *In the logarithmic spiral*

$$\varrho = a^\theta;$$

$$\therefore \frac{d\varrho}{d\theta} = la \times a^\theta \text{ (Art. 17.)}$$

$$\therefore -\frac{d^2\varrho}{d\theta^2} = -(la)^2 \times a^\theta = -(la)^2 . \varrho$$

$$\therefore R = \frac{(\varrho^2 + (la)^2\varrho^2)^{\frac{3}{2}}}{2(la)^2\varrho^2 - (la)^2\varrho^2 + \varrho^2} = \frac{\varrho^3(1 + (la)^2)^{\frac{3}{2}}}{\varrho^2(1 + (la)^2)}$$
$$= \varrho\{1 + (la)^2\}^{\frac{1}{2}}$$

Ex. 2. *In the spiral of Archimedes*

$$\varrho = a\,\theta$$

and

$$R = \frac{(\varrho^2 + a^2)^{\frac{3}{2}}}{2\,a^2 + \varrho^2}.$$

Ex. 3. *In the hyperbolic spiral*

$$\varrho = \frac{a}{\theta}$$

$$\therefore R = \frac{\varrho\,(a^2 + \varrho^2)^{\frac{3}{2}}}{a^3}.$$

Ex. 4. *In the Lituus*

$$\varrho^2 = \frac{a^2}{\theta}$$

$$\therefore R = \frac{\varrho}{2a^2} \cdot \frac{(4\,a^4 + \varrho^4)^{\frac{3}{2}}}{4\,a^4 - \varrho^4}.$$

Ex. 5. *In the Epicycloid*

$$\varrho = (r + r')^2 - 2\,r\,(r + r')\cos.\theta$$

r and r′ being the radius of the wheel and globe respectively.

Here

$$R = \frac{(r + r')\,(3\,r^2 - 2\,r\,r' - r'^2 + 2\,\varrho)^{\frac{3}{2}}}{2\,(3\,r^2 - 2\,r\,r' - r'^2) + 3\,\varrho}.$$

Having already given those results of the Calculus of Indefinite Differences which are most useful, we proceed to the reverse of the calculus, which consists in the investigation of the Indefinites themselves from their indefinite differences. In the direct method we seek the Indefinite Difference of a given function. In the inverse method we have given the *Indefinite Difference* to find the function whose Indefinite Difference it is. This inverse method we call

THE INTEGRAL CALCULUS

OR

INDEFINITE DIFFERENCES.

76. The integral of d x is evidently x + C, since the indefinite difference of x + C is d x.

77. *Required the integral of* a d x?

By Art. 9, we have

$$d\,(a\,x) = a\,d\,x.$$

Hence reversely the integral of a d x is a x. This is only one of the innumerable integrals which there are of a d x. We have not only d (a x) = a d x but also

$$d(ax + C) = a\,dx$$

in which C is any constant whatever.

$$\therefore ax + C = \int a\,dx = a\int dx \quad . \quad . \quad . \quad (a) \quad \text{(see 76)}$$

generally, $\int$ being the characteristic of an integral.

78. *Required the integral of*

$$a\,x^p\,dx.$$

By Art. 12

$$d(a\,x^n + C) = n\,a\,x^{n-1}dx$$
$$\therefore a\,x^n + C = \int n\,a\,x^{n-1}dx$$
$$= n \times \int a\,x^{n-1}dx \;(77)$$
$$\therefore \int a\,x^{n-1}dx = \frac{a\,x^n}{n} + \frac{C}{n}.$$

But since C is any constant whatever $\frac{C}{n}$ may be written C.

$$\therefore \int a\,x^{n-1}dx = \frac{a\,x^n}{n} + C$$

Hence it is plain that

$$\int a\,x^p\,dx = \frac{a\,x^{p+1}}{p+1} + C$$

Or *To find the integral of the product of a constant the* p[th] *power of the variable and the Indefinite Difference of that variable, let the index of the power be increased by* 1, *suppress the Indefinite Difference, multiply by the constant, divide by the increased index, and add an arbitrary constant.*

79. Hence

$$\int (a\,x^p\,dx + b\,x^q\,dx + \&c.) =$$
$$\frac{a\,x^{p+1}}{p+1} + \frac{b\,x^{q+1}}{q+1} + \&c. + C$$

80. Hence also

$$\int a\,x^{-n}\,dx = -\frac{a}{(n-1)\,x^{n-1}} + C.$$

81. *Required the integral of*

$$a\,x^{m-1}\,dx\,(b + e\,x^m)^p.$$

Let

$$u = b + e\,x^m$$
$$\therefore du = m\,e\,x^{m-1}\,dx$$
$$\therefore a\,x^{m-1}\,dx = \frac{a}{m\,e}.\,du$$
$$\therefore \int a\,x^{m-1}dx(b + e\,x^m)^p = \int \frac{a}{m\,e}\,u^p\,du$$

$$= \frac{a}{m\,e\,.\,(p+1)} \cdot u^{p+1} + C \ (78)$$

$$= \frac{a}{m\,e\,(p+1)} \cdot (b \times e\,x^m)^{p+1} + C.$$

82. *Required the integral of* $\frac{d\,x}{x}$.

By 80 it would seem that

$$\int \frac{d\,x}{x} = -\frac{1}{0} + C$$

and if when

$$\int \frac{d\,x}{x} = 0,\ C = c,\ w - \int \frac{d\,x}{x} = \frac{1}{0} - \frac{1}{0} = \frac{0}{0}.$$

But by Art. 17 a. we know that

$$d\,.\,l\,x = \frac{d\,x}{x}$$

Therefore

$$\int \frac{d\,x}{x} = l\,x + C.$$

Here it may be convenient to make the arbitrary constant of the form l C

Therefore

$$\int \frac{d\,x}{x} = l\,x + l\,C = l\,C\,x$$

Hence the integral of a fraction whose numerator is the Indefinite Difference of the denominator, is the hyperbolic logarithm of the denominator PLUS *an arbitrary constant.*

83. Hence

$$\int \frac{a\,x^{m-1}\,d\,x}{b\,x^m + e} = \frac{a}{b\,m} \int \frac{m\,x^{m-1}\,d\,x}{x^m + \frac{e}{b}}$$

$$= \frac{a}{b\,m} . l . \left(x^m + \frac{e}{b}\right) + C$$

$$= \frac{a}{b\,m} . l . C\left(x^m + \frac{e}{b}\right),$$

and so on for more complicated forms.

84. *Required the integral of* $a^x\,d\,x$.

By Art. 17

$$d\,.\,a^x = l\,a\,.\,a^x\,d\,x$$

$$\therefore \int a^x\,d\,x = \frac{1}{l\,a} . \int d\,a^x$$

$$= \frac{1}{l\,a} . a^x + C$$

85. If y, x, t, s denote the sine, cosine, tangent, and secant of an angle θ; then we have, Art. 26, 27.

$$d\theta = \frac{dy}{\sqrt{(1-y^2)}} = \frac{-dx}{\sqrt{(1-x^2)}} = \frac{dt}{1+t^2} = \frac{ds}{s\sqrt{2s-s^2}}$$

$$\therefore \int \frac{dy}{\sqrt{(1-y^2)}} = \theta + C = \sin.^{-1} y + C$$

$$\int \frac{-dx}{\sqrt{(1-x^2)}} = \theta + C = \cos.^{-1} x + C$$

$$\int \frac{dt}{1+t^2} = \theta + C = \tan.^{-1} t + C$$

$$\int \frac{ds}{s\sqrt{2s-s^2}} = \theta + C = \sec.^{-1} s + C$$

$\sin.^{-1} y$, $\cos.^{-1} x$, &c. being symbols for the arc whose sine is y, cosine is x, &c. respectively.

86. Hence, more generally,

$$\int \frac{du}{\sqrt{(a-bu^2)}} = \frac{1}{\sqrt{b}} \int \frac{\sqrt{\frac{b}{a}}\,du}{\sqrt{\left(1-\frac{b}{a}u^2\right)}}$$

$$= \frac{1}{\sqrt{b}} \cdot \sin.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (a)$$

or $= \frac{1}{\sqrt{b}} \times$ angle whose sine is $u\sqrt{\frac{b}{a}}$ to rad. $1 + C$.

Also

$$\int \frac{-du}{\sqrt{(a-bu^2)}} = \frac{1}{\sqrt{b}} \cdot \cos.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (b)$$

Again

$$\int \frac{du}{a+bu^2} = \frac{1}{\sqrt{ab}} \cdot \int \frac{\sqrt{\frac{b}{a}}\,du}{1+\frac{b}{a}u^2}$$

$$= \frac{1}{\sqrt{ab}} \tan.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (c)$$

and

$$\int \frac{du}{u\sqrt{(bu^2-a)}} = \frac{1}{\sqrt{a}} \int \frac{\sqrt{\frac{b}{a}}\,du}{u\sqrt{\frac{b}{a}} \times \sqrt{\left(\frac{b}{a}u^2-1\right)}}$$

$$= \frac{1}{\sqrt{a}} \cdot \sec.^{-1} u\sqrt{\frac{b}{a}} + C \quad . \quad . \quad (d)$$

Moreover, if u be the versed sine of an angle θ, then the sine $= \sqrt{(2u - u^2)}$ and

$$du = d(1 - \cos.\theta) = d\theta.\sin.\theta \text{ (Art. 27.)}$$
$$= d\theta.\sqrt{(2u - u^2)}$$
$$\therefore d\theta = \frac{du}{\sqrt{(2u - u^2)}}$$

Hence

$$\int \frac{du}{\sqrt{(2u - u^2)}} = \theta + C$$
$$= \text{vers.}^{-1} u + C$$

and generally

$$\int \frac{du}{\sqrt{(au - bu^2)}} = \int \frac{\frac{2b}{a} du}{\sqrt{b} \sqrt{\left(2.\frac{2b}{a}u - \left(\frac{2b}{a}\right)^2 u^2\right)}}$$
$$= \frac{1}{\sqrt{b}} \text{vers.}^{-1} \frac{2b}{a} u + C \quad . \; . \; . \; . \quad (e)$$

87. *Required the integrals of*

$$\frac{dx}{a + bx}, \quad \frac{dx}{a - bx}, \quad \frac{dx}{a - bx^2}.$$

$$\int \frac{dx}{a + bx} = \frac{1}{b}.\int \frac{d.(a + bx)}{a + bx}$$
$$= \frac{1}{b}.\,l.\,(a + bx) + C \quad . \; . \; . \; . \; . \; . \quad (f)$$

and

$$\int \frac{dx}{a - bx} = -\frac{1}{b}\int \frac{d(a - bx)}{a - bx}$$
$$= -\frac{1}{b}.\,l.\,(a - bx) + C \quad . \; . \; . \quad (g)$$

see Art. 17 a.

Hence,

$$\int dx \left\{\frac{1}{a + bx} + \frac{1}{a - bx}\right\} = \int \frac{2a\,dx}{a^2 - b^2x^2}$$
$$= \frac{1}{b}.\,l.(a + bx) - \frac{1}{b}.\,l.\,(a - bx) + C$$
$$= \frac{1}{b}.\,l.\,\frac{a + bx}{a - bx} + C.$$

Hence we easily get by analogy

$$\left.\begin{aligned}\int\frac{dx}{a-bx^2} &= \frac{1}{\sqrt{ab}}\cdot l.\frac{\sqrt{a}+\sqrt{b}.x}{\sqrt{a}-bx^2}+C\\ &= \frac{1}{2\sqrt{ab}}\cdot l.\frac{\sqrt{a}+\sqrt{b}.x}{\sqrt{a}-\sqrt{b}.x}+C\end{aligned}\right\}\ldots(h)$$

88. *Required the integral of*

$$\frac{dx}{ax^2+bx+c}.$$

In the first place

$$ax^2+bx+c = a\left\{x+\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a}\right\}\times$$

$$\left\{x+\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}\right\} = a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{2a}\right\}$$

Hence, putting

$$x+\frac{b}{2a}=u$$

we have

$$dx = du$$

and

$$\frac{dx}{ax^2+bx+c} = \frac{du}{a\left(u^2-\frac{b^2-4ac}{2a}\right)}$$

which presents the following cases.

Case 1. *Let* a *be negative and* c *be positive; then*

$$\frac{dx}{-ax^2+bx+c} = \frac{du}{-a\left(\frac{b^2+4ac}{2a}+u^2\right)}$$

$$\therefore\int\frac{dx}{-ax^2+bx+c} = \frac{\sqrt{2}}{-\sqrt{a}\sqrt{(b^2+4ac)}}\tan.^{-1}u\sqrt{\frac{2a}{b^2+4ac}}+C$$

$$\text{(see Art. 86)} = -\sqrt{\frac{2}{a(b^2+4ac)}}\cdot\tan.^{-1}\left(x+\frac{b}{2a}\right)\sqrt{\frac{2a}{b^2+4ac}}+C\ldots(i)$$

Case 2. *Let* c *be negative and* a *positive; then*

$$\int\frac{dx}{ax^2+bx-c} = \int\frac{du}{a\left(u^2-\frac{b^2+4ac}{2a}\right)}$$

$$= -\frac{1}{a}\int\frac{du}{\frac{b^2+4ac}{2a}-u^2}$$

$$= -\sqrt{\frac{1}{2a(b^2+4ac)}}.l.\frac{\sqrt{\frac{b^2+4ac}{2a}}+x+\frac{b}{2a}}{\sqrt{\frac{b^2+4ac}{2a}}-x-\frac{b}{2a}}+C\ldots(k)$$

see Art. 87.

Case 3. *Let* b^2 *be* $>$ $4ac$ *and* a, c *be both positive; then*

$$\int \frac{dx}{ax^2+bx+c} = \int \frac{du}{a\left(u^2 - \frac{b^2-4ac}{2a}\right)}$$

$$= \frac{1}{-a}\int \frac{du}{\frac{b^2-4ac}{2a} - u^2}$$

$$= -\sqrt{\frac{1}{2a(b^2-4ac)}}.\mathrm{l}.\frac{\sqrt{\frac{b^2-4ac}{2a}}+x+\frac{b}{2a}}{\sqrt{\frac{b^2-4ac}{2a}}-x-\frac{b}{2a}}+C \quad (\mathrm{l})$$

Case 4. *Let* b^2 *be* $<$ $4ac$ *and* a, c *be both positive;*
Then

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{du}{\frac{4ac-b^2}{2a}+u^2}$$

$$= \sqrt{\frac{2}{a(4ac-b^2)}}.\tan.^{-1}\left(x \pm \frac{b}{2a}\right)\sqrt{\frac{2a}{4ac-b^2}}+C \quad (\mathrm{m})$$

Case 5. *If* b^2 *be* $>$ $4ac$ *and* a, c *both negative;*
Then

$$\int \frac{dx}{-ax^2+bx-c} = \frac{1}{-a}\int \frac{du}{\frac{b^2-4ac}{2a}+u^2}$$

$$= -\sqrt{\frac{2}{a(b^2-4ac)}}\tan.^{-1}\left(x+\frac{b}{2a}\right)\sqrt{\frac{2a}{b^2-4ac}}+C \quad (\mathrm{n})$$

Case 6. *If* b^2 *be* $<$ $4ac$ *and* a *and* c *both negative;*
Then

$$\int \frac{dx}{-ax^2+bx-c} = \frac{1}{a}\int \frac{du}{\frac{4ac-b^2}{2a}-u^2}$$

$$= \sqrt{\frac{1}{2a(4ac-b^2)}}.\mathrm{l}.\frac{\sqrt{\frac{4ac-b^2}{2a}}+x+\frac{b}{2a}}{\sqrt{\frac{4ac-b^2}{2a}}-x-\frac{b}{2a}}+C \quad (\mathrm{o})$$

89. *Required the integral of any rational function whatever of one variable, multiplied by the indefinite difference of that variable.*

Every rational function of x is comprised under the general form

$$\frac{Ax^m + Bx^{m-1} + Cx^{m-2} + \&c.\ Kx + L}{ax^n + bx^{n-1} + cx^{n-2} + \&c.\ kx + l}$$

in which A, B, C, &c. a, b, c, &c. and m, n are any constants whatever.

If

$$n = 0,$$

then we have (Art. 77)

$$\int (A x^m + B x^{m-1} + \&c.) \frac{dx}{a} = \left(\frac{A x^{m+1}}{m+1} + \frac{B x^m}{m}\right.$$

$$\left. + \frac{C x^{m-1}}{m-1} + \&c.\right) \frac{1}{a} + \text{constant.}$$

Again, if m be > n the above can always be reduced by actual division to the form

$$A' x^{m-n} + B' x^{m-n-1} + \&c. + \frac{A'' x^{n-1} + B'' x^{n-2} + \&c.}{a x^n + b x^{n-1} + \&c.}$$

and if the whole be multiplied by d x its integral will consist of two parts, one of which is found to be (by 77)

$$\frac{A'}{m-n+1} \cdot x^{m-n+1} + \frac{B' . x^{m-n}}{m-n} + \&c.$$

and the other

$$\int \frac{A'' x^{n-1} + B'' x^{n-2} + \&c.}{a x^n + b x^{n-1} + \&c.} d x.$$

Hence then it is necessary to consider only functions of the general form

$$\frac{x^{n-1} + A x^{n-2} + B x^{n-3} + \&c.}{x^n + a x^{n-1} + b x^{n-2} + \&c.} = \frac{U}{V}$$

in order to integrate an indefinite difference, whose definite part is any rational function whatever.

Case 1. *Let the denominator* V *consist of* n *unequal real factors,* $x - \alpha$, $x - \beta$, &c. *according to the theory of algebraic equations.* Assume

$$\frac{U}{V} = \frac{P}{x-\alpha} + \frac{Q}{x-\beta} + \frac{R}{x-\gamma} + \&c.$$

and reducing to a common denominator we shall have

$$\begin{aligned} U = {} & P . \overline{x-\beta} . \overline{x-\gamma} \ldots \text{to } (n-1) \text{ terms} \\ & + Q . \overline{x-\alpha} . \overline{x-\gamma} \ldots\ldots\ldots \\ & + R . \overline{x-\alpha} . \overline{x-\beta} \ldots\ldots\ldots \\ = {} & (P + Q + R + \&c.) x^{n-1} \\ & - \{P . (\underset{1}{S} - \alpha) + Q . (\underset{1}{S} - \beta) + \&c.\} x^{n-2} \\ & + \{P . (\underset{1.2}{S} - \alpha . \overline{\underset{1}{S} - \alpha}) + Q . (\underset{1.2}{S} - \beta . \overline{\underset{1}{S} - \beta}) + \&c.\} x^{n-3} \\ & - \&c. \end{aligned}$$

where $\underset{1}{S}$, $\underset{1.2}{S}$ &c. denote the sum of α, β, γ &c. the sum of the products of every two of them and so on.

But by the theory of equations

$$\underset{1}{S} = -a$$
$$\underset{1.2}{S} = b$$
$$\&c. = \&c.$$

$$\therefore U = (P + Q + R + \&c.)\, x^{n-1}$$
$$+ \{a(P + Q + R + \&c.) + P\alpha + Q\beta + R\gamma + \&c.\} \times x^{n-2}$$
$$+ \{b(P + Q + R + \&c.) + a(P\alpha + Q\beta + \&c.) +$$
$$(P\alpha^2 + Q\beta^2 + R\gamma^2 + \&c.)\}\, x^{n-3} + \&c.$$

Hence equating like quantities (6)

$$P + Q + R + \&c. = 1$$
$$a + P\alpha + Q\beta + R\gamma + \&c. = A$$
$$b + a(A - a) + P\alpha^2 + Q\beta^2 + R\gamma^2 + \&c. = B$$
$$\&c. = \&c.$$

giving n independent equations to determine P, Q, R, &c.

Ex. 1. Let $\frac{U}{V} = \frac{x^2 + 6x + 3}{x^3 + 6x^2 + 11x + 6}$

Here

$$\left.\begin{array}{r} P + \quad Q + \quad R = 1 \\ 6 + P + 2Q + 3R = 6 \\ 11 + P + 4Q + 9R = 3 \end{array}\right\}\text{whence}$$
$$P = -1,\ Q = 5 \text{ and } R = -3$$

Hence

$$\int \frac{U\,dx}{V} = \int \frac{-dx}{x+1} + \int \frac{5\,dx}{x+2} - \int \frac{3\,dx}{x+3}$$
$$= C - \mathrm{l.}(x + 1) + 5\,\mathrm{l.}(x + 2) - 3\,\mathrm{l.}(x + 3).$$

P, Q, R, &c. may be more easily found as follows:
Since

$$x^{n-1} + A\,x^{n-2}\,\&c. = P(x - \beta).(x - \gamma).\&c.$$
$$+ Q(x - \alpha).(x - \gamma).\&c.$$
$$+ R(x - \alpha).(x - \beta).\&c.$$
$$+ \&c.$$

let $x = \alpha, \beta, \gamma$, &c. successively; we shall then have

$$\left.\begin{array}{l} \alpha^{n-1} + A\alpha^{n-2} + \&c. = P.(\alpha - \beta).(\alpha - \gamma)\,\&c. \\ \beta^{n-1} + A\beta^{n-2} + \&c. = Q.(\beta - \alpha).(\beta - \gamma)\,\&c. \\ \gamma^{n-1} + A\gamma^{n-2} + \&c. = R.(\gamma - \alpha).(\gamma - \beta)\,\&c. \end{array}\right\}\ldots (A)$$
$$\&c. = \&c.$$

In the above example we have

$$\alpha = -1,\ \beta = -2,\ \gamma = -3 \text{ and } n = 3$$
$$A = 6 \text{ and } B = 3.$$

$$\therefore P = \frac{1 - 6 + 3}{1.\ 2.} = -1$$

$$Q = \frac{4 - 6.\ 2 + 3}{-1.\ 1} = 5$$

$$R = \frac{9 - 6.\ 3 + 3}{-2 \times -1} = -3$$

as before.

Hence then the factors of V being supposed all unequal, either of the above methods will give the coefficients P, Q, R, &c. and therefore enable us to analyze the general expression $\frac{U}{V}$ into the *partial* fractions as expressed by

$$\frac{U}{V} = \frac{P}{x - \alpha} + \frac{Q}{x - \beta} + \&c.$$

and we then have

$$\int \frac{U\,d\,x}{V} = P \,.\, l\,(x - \alpha) + Q\,l.\,(x - \beta) + \&c. + C.$$

Ex. 2. $\int \frac{a^3 + b\,x^2}{a^2 x - x^3} = \int \frac{a\,d\,x}{x} + \frac{a+b}{2} \int \frac{d\,x}{a - x} - \frac{a+b}{2} \int \frac{d\,x}{a + x}$

$$= a\,l\,x - \frac{a+b}{2}\,l\,(a - x) - \frac{a+b}{2}\,l\,.\,(a + x) + C$$

$$= a\,l\,x - (a + b)\,l\,\sqrt{a^2 - x^2} + C$$

by the nature of logarithms.

Ex. 3. $\int \frac{3\,x - 5}{x^2 - 6\,x + 8}\,d\,x = -\frac{1}{2} \int \frac{d\,x}{x - 2} + \frac{7}{2} \int \frac{d\,x}{x - 4} + C$

$= \frac{7}{2}\,.\,l\,(x - 4) - \frac{1}{2}\,l.\,(x - 2) + C.$

Ex. 4. $\int \frac{x\,d\,x}{x^2 + 4\,a\,x - b^2} = \int \frac{P\,d\,x}{x + \alpha} + \int \frac{Q\,d\,x}{x + \beta} = P\,l\,(x + \alpha)$
$+ Q\,l\,.\,(x + \beta) + C$

where

$$\alpha = 2\,a + \sqrt{(4\,a^2 + b^2)},\ \beta = 2\,a - \sqrt{(4\,a^2 + b^2)}$$

and

$$P = \frac{\alpha}{\alpha - \beta} = \frac{2\,a + \sqrt{(4\,a^2 + b^2)}}{2\sqrt{(4\,a^2 + b^2)}}$$

$$Q = \frac{-\beta}{\alpha - \beta} = \frac{\sqrt{(4\,a^2 + b^2)} - 2\,a}{2\sqrt{(4\,a^2 + b^2)}}$$

Case 2. *Let all the factors of* V *be real and equal, or suppose* $\alpha = \beta = \gamma = \&c.$

Then

$$\frac{U}{V} = \frac{x^{n-1} + A\,x^{n-2} + \&c.}{(x - \alpha)^n}.$$

and since

$$\alpha - \beta = 0, \alpha - \gamma = 0 \text{ \&c.}$$

the forms marked (A) will not give us P, Q, R, &c. In this case we must assume

$$\frac{U}{V} = \frac{P}{(x-\alpha)^n} + \frac{Q}{(x-\alpha)^{n-1}} + \frac{R}{(x-\alpha)^{n-2}} + \text{\&c.}$$

to $\overline{n-1}$ terms, and reducing to a common denominator, we get

$$U = P + Q.(x-\alpha) + R(x-\alpha)^2 + \text{\&c.}$$

now let $x = \alpha$, and we have

$$\alpha^{n-1} + A\alpha^{n-2} + \text{\&c.} = P.$$

Also

$$\frac{dU}{dx} = Q + 2R.(x-\alpha) + 3S.(x-\alpha)^2 + \text{\&c.}$$

$$\frac{d^2U}{dx^2} = 2R + 3.2.S.(x-\alpha) + 4.3.T(x-\alpha)^2 + \text{\&c.}$$

$$\frac{d^3U}{dx^3} = 2.3.S + 4.3.2T(x-\alpha) + \text{\&c.}$$

$$\text{\&c.} = \text{\&c.}$$

and if in each of these x be put $= \alpha$, we have by Maclaurin's theorem the values of Q, R, S, &c.

Ex. 1. Let $\frac{U}{V} = \frac{x^2 - 3x + 2}{(x-4)^3}$.

Then

$$U = x^2 - 3x + 2$$

$$\frac{dU}{dx} = 2x - 3$$

$$\frac{d^2U}{dx^2} = 2$$

$$\therefore P = 6$$

$$Q = 8 - 3 = 5$$

$$R = \tfrac{1}{2}.2 = 1$$

$$\therefore \int \frac{U\,dx}{V} = \int \frac{6\,dx}{(x-4)^3} + \int \frac{5\,dx}{(x-4)^2} + \int \frac{dx}{x-4}$$

$$= C - 3\frac{1}{(x-4)^2} - \frac{5}{x-4} + 1(x-4)$$

$$= C + \frac{17-5x}{(x-4)^2} + 1(x-4).$$

Ex. 2. Let $\frac{U}{V} = \frac{x^5 + x^3}{(x-3)^6}$.

Here

$$U = x^5 + x^3$$

$$\frac{dU}{dx} = 5x^4 + 3x^2$$

$$\frac{d^2U}{dx^2} = 20x^3 + 6x$$

$$\frac{d^3U}{dx^3} = 60x^2 + 6$$

$$\frac{d^4U}{dx^4} = 120x$$

$$\frac{d^5U}{dx^5} = 120$$

$$\therefore P = 3^5 + 3^3 = 27 \times 10 = 270$$

$$Q = 27 \times 16 = 432$$

$$R = \frac{20 \times 27 + 6 \times 3}{2} = 279$$

$$S = \frac{9 \times 60 + 6}{2 \times 3} = 91$$

$$T = \frac{360}{2.3.4.} = 15$$

$$W = \frac{120}{2.3.4.5} = 1.$$

Hence

$$\int \frac{x^5 + x^3}{(x-3)^6} dx = 270 \int \frac{dx}{(x-3)^6} + 432 \int \frac{dx}{(x-3)^5} + 279 \int \frac{dx}{(x-3)^4}$$

$$+ 91 \int \frac{dx}{(x-3)^3} + 15 \int \frac{dx}{(x-3)^2} + \int \frac{dx}{x-3} + C = C - 54 . \frac{1}{(x-3)^5}$$

$$- 108 . \frac{1}{(x-3)^4} - 93 . \frac{1}{(x-3)^3} - \frac{91}{2} . \frac{1}{(x-3)^2} - 15 . \frac{1}{x-3} + 1 . (x-3)$$

which admits of farther reduction.

Ex. 3. Let $\frac{x^2 + x}{(x-1)^5} = \frac{U}{V}$.

Here

$$U = x^2 + x$$

$$\frac{dU}{dx} = 2x + 1$$

and

$$\frac{d^2U}{dx^2} = 2.$$

Hence

$$P = 2$$
$$Q = 3$$
$$R = \frac{2}{2} = 1$$

$$\therefore \int \frac{x^2 + x}{(x-1)^5} dx = 2\int \frac{dx}{(x-1)^5} + 3\int \frac{dx}{(x-1)^4} + \int \frac{dx}{(x-1)^3}$$
$$= C - \frac{1}{2} \cdot \frac{1}{(x-1)^4} - \frac{1}{(x-1)^3} - \frac{1}{2} \cdot \frac{1}{(x-1)^2}$$
$$= C - \frac{x^2}{2(x-1)^4}$$

It appears from this example, and indeed is otherwise evident, that the number of partial fractions into which it is necessary to split the function exceeds the dimension of x *in* U, *by unity.*

This is the first time, unless we mistake, that *Maclaurin's Theorem* has been used to analyze rational fractions into partial rational fractions. It produces them with less labour than any other method that has fallen under our notice.

Case 3. *Let the factors of the denominator* V *be all imaginary and unequal.*

We know then if in V, which is real, there is an imaginary factor of the form $x + h + k\sqrt{-1}$, then there is also another of the form $x + h - k\sqrt{-1}$. Hence V must be of an even number of dimensions, and must consist of quadratic real factors of the form arising from

$$(x + h + k\sqrt{-1})(x + h - k\sqrt{-1})$$

or of the form

$$(x + h)^2 + k^2.$$

Hence, assuming

$$\frac{U}{V} = \frac{P + Qx}{(x + \alpha)^2 + \beta^2} + \frac{P' + Q'x}{(x + \alpha')^2 + \beta'^2} + \&c.$$

and reducing to a common denominator, we have

$$\begin{aligned} U = &(P + Qx)\{(x + \alpha')^2 + \beta'^2\}\{(x + a'')^2 + \beta''^2\} \times \&c. \\ &+ (P' + Q'x)\{(x + \alpha)^2 + \beta^2\}\{(x + \alpha'')^2 + \beta''^2\} \times \&c. \\ &+ (P'' + Q''x)\{(x + \alpha)^2 + \beta^2\}\{(x + \alpha')^2 + \beta'^2\} \times \&c. \\ &+ \&c. \end{aligned}$$

Now for x substitute successively

$$\alpha + \beta\sqrt{-1},\ \alpha' + \beta'\sqrt{-1},\ \alpha'' + \beta''\sqrt{-1}, \&c.$$

then U will become for each partly real and partly imaginary, and we have as many equations containing respectively P, Q; P′, Q′; P″, Q″, &c. as there are pairs of these coefficients; whence by equating *homogeneous* quantities, viz. real and imaginary ones, we shall obtain P, Q; P′, Q′, &c.

Ex. 1. *Required the integral of*

$$\frac{x^3 d x}{x^4 + 3 x^2 + 2}.$$

Here the quadratic factors of V are $x^2 + 1$, $x^2 + 2$

$$\therefore \alpha = 0,\ \alpha' = 0,\ \beta = 1, \text{ and } \beta' = \sqrt{2}.$$

Consequently

$$x^3 = (P + Q x)(x^2 + 2) + (P' + Q' x)(x^2 + 1)$$

Let $x = \sqrt{-1}$. Then

$$-\sqrt{-1} = (P + Q\sqrt{-1}).(-1 + 2) = P + Q\sqrt{-1}$$

$$\therefore P = 0,\ Q = -1$$

Again, let $x = \sqrt{2}.\sqrt{-1}$, and we have

$$-2^{\frac{3}{2}}\sqrt{-1} = (P' + Q'\sqrt{2}.\sqrt{-1})(-2 + 1) = -P' - Q'\sqrt{2}.\sqrt{-1}$$

$$\therefore P' = 0, \text{ and } Q' = 2$$

Hence

$$\int\frac{x^3 d x}{x^4 + 3 x^2 + 2} = \int\frac{-x d x}{x^2 + 1} + \int\frac{2 x d x}{x^2 + 2} = C - \tfrac{1}{2}\, l\,(x^2 + 1) + l.\,(x^2 + 2)$$

Ex. 2. *Required the integral of*

$$\frac{d x}{1 + x^{2n}}.$$

To find the quadratic factors of

$$1 + x^{2n}$$

we assume

$$x^{2n} + 1 = 0,$$

and then we have

$$x^{2n} = -1 = \cos.(2p + 1)\pi + \sqrt{-1}\sin.(2p + 1)\pi$$

π being 180° of the circle whose diameter is 1, and p any integer whatever.

Hence by Demoivre's Theorem

$$x = \cos.\frac{2p + 1}{2n}\pi + \sqrt{-1}.\sin.\frac{2p + 1}{2n}\pi$$

But since imaginary roots of an equation enter it by pairs of the form $A \pm \sqrt{-1}.B$, we have also

$$x = \cos.\frac{2p + 1}{2n}\pi - \sqrt{-1}.\sin.\frac{2p + 1}{2n}\pi$$

and

$$\therefore \left(x - \cos.\frac{2p+1}{2n}\pi - \sqrt{-1}.\sin.\frac{2p+1}{2n}\pi\right) \times$$
$$\left(x - \cos.\frac{2p+1}{2n}\pi + \sqrt{-1}\sin.\frac{2p+1}{2n}\pi\right) =$$
$$x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1$$

which is the general quadratic factor of $x^{2n} + 1$. Hence putting $p = 0, 1, 2 \ldots\ldots n-1$ successively,

$$x^{2n} + 1 = \left(x^2 - 2x\cos.\frac{\pi}{2n} + 1\right).\left(x^2 - 2x\cos.\frac{3\pi}{2n} + 1\right) \times$$
$$\left(x^2 - 2x\cos.\frac{5\pi}{2n} + 1\right) \times \ldots\ldots \left(x^2 - 2x\cos.\frac{2n-1}{2n} + 1\right).$$

Hence to get the values of P and Q corresponding to the general factor, assume

$$\frac{1}{1+x^{2n}} = \frac{P + Qx}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1} + \frac{N}{M}.$$

Then

$$1 = (P + Qx).M + N\left(x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1\right).$$

But

$$M = \frac{1+x^{2n}}{x^2 - 2x\cos.\frac{2p+1}{2n}\pi + 1}$$

and becomes of the form $\frac{0}{0}$ when for x we put $\cos.\frac{2p+1}{2n}\pi + \sqrt{-1} \times \sin.\frac{2p+1}{2n}\pi$; its value however may thus be found

Let $\cos.\frac{2p+1}{2n}\pi + \sqrt{-1}\sin.\frac{2p+1}{2n}\pi = r$

then

$$\cos.\frac{2p+1}{2n}\pi - \sqrt{-1}.\sin.\frac{2p+1}{2n}\pi = \frac{1}{r}$$

and

$$M = \frac{1 + x^{2n}}{(x - r.\left(x - \frac{1}{r}\right)}$$

Again let $x - r = y$; then

$$M = \frac{1 + y^{2n} + 2n\,y^{2n-1}r + \&c.\ldots.2n\,y\,r^{2n-1} + r^{2n}}{y\left(x - \frac{1}{r}\right)}$$

But

$$r^{2n} = \cos.\overline{2p+1}.\pi + \sqrt{-1}\sin.\overline{2p+1}.\pi = -1$$

$$\therefore M = \frac{y^{2n-1} + 2n\,y^{2n-2}.r + \ldots.2n\,r^{2n-1}}{x - \frac{1}{r}}.$$

Hence when for x we put r, $y = 0$, and

$$M = \frac{2n\ r^{2n-1}}{r - \frac{1}{r}}$$

and from the above equation we have

$$1 = (P + Q\,r)\,\frac{2n\ r^{2n-1}}{r - \frac{1}{r}}.$$

or

$$2\sqrt{-1}\sin.\frac{2p+1}{2n}\pi = 2nP.\cos.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi + 2nP\sqrt{-1}\times$$

$$\sin.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi - 2nQ \text{ (since } r^{2n} = -1)$$

$\therefore$ equating *homogeneous quantities* we get

$$\sin.\frac{2p+1}{2n}\pi = nP.\sin.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi$$

and

$$P.\cos.\frac{\overline{2p+1}.\overline{2n-1}}{2n}\pi = Q.$$

But

$$\frac{\overline{2p+1}.\overline{2n-1}.\pi}{2n} = \overline{2p+1}.\pi - \frac{2p+1}{2n}\pi$$

Hence the above equations become

$$\therefore \sin.\frac{2p+1}{2n}\pi = n\,P\sin.\frac{2p+1}{2n}$$

$$-P\cos.\frac{2p+1}{2n}\pi = Q$$

$$\therefore P = \frac{1}{n},\ \text{and}\ Q = -\frac{1}{n}.\cos.\frac{2p+1}{2n}\pi.$$

Hence the general partial integral of

$$\frac{dx}{1+x^{2n}}\ \text{is}$$

$$\frac{1}{n}\int\frac{\left(1-x\cos.\frac{2p+1}{2n}\pi\right)dx}{x^2-2x\cos.\frac{2p+1}{2n}\pi+1}=$$

$$\frac{-\cos.\frac{2p+1}{2n}\pi}{2n}\cdot\int\frac{2x\ dx-2\cos.\frac{2p+1}{2n}\pi.dx}{x^2-2x\cos.\frac{2p+1}{2n}\pi+1}+$$

$$\frac{\sin.^2\frac{2p+1}{2n}\pi}{n}\int\frac{dx}{x^2-2x\cos.\frac{2p+1}{2n}\pi+1}=$$

$$C-\frac{\cos.\frac{2p+1}{2n}\pi}{2n}.l\left(x^2-2x\cos.\frac{2p+1}{2n}\pi+1\right)$$

$$+\frac{\sin.\frac{2p+1}{2n}\pi}{n}\times\tan.^{-1}\left(\frac{x-\cos.\frac{2p+1}{2n}\pi}{\sin.\frac{2p+1}{2n}\pi}\right)$$

see Art. 88. Case 4.

Hence then the integral of $\frac{dx}{1+x^n}$, which is the aggregate of the results obtained from the above general form by substituting for $p = 0, 1, 2 \ldots n-1$, may readily be ascertained.

As a particular instance let $\int\frac{dx}{1+x^6}$ *be required.*

Here

$$n = 3$$

and the general term is

$$\frac{-\cos.\frac{2p+1}{6}\pi}{6}.l.\left(x^2-2x\cos.\frac{2p+1}{6}\pi+1\right)$$

$$+\frac{\sin.\frac{2p+1}{6}\pi}{3}.\tan.^{-1}\frac{x-\cos.\frac{2p+1}{6}\pi}{\sin.\frac{2p+1}{6}\pi}$$

Let $p=0, 1, 2$, collect the terms, and reduce them; and it will appear that

$$\int\frac{dx}{1+x^6}=\frac{1}{6}\left\{\frac{\sqrt{3}}{2}.l.\frac{x^2+x\sqrt{3}+1}{x^2-x\sqrt{3}+1}+\tan.^{-1}\frac{3x(1-x^2)}{x^4-4x^2+1}\right\}+C.$$

By proceeding according to the above method it will be found, that the general partial fractions to be integrated in the integrals of

$$\frac{dx}{x^n - 1} \text{ and } \frac{x^r dx}{x^n - 1}$$

are respectively

$$\frac{2}{n} \cdot \frac{\cos. \frac{2p\pi}{n} x - 1}{x^2 - 2x \cos. \frac{2p\pi}{n} + 1} \cdot dx$$

and

$$\frac{2}{n} \times \frac{\cos. \frac{(r+1).2p\pi}{n} x - \cos. \frac{2rp\pi}{n}}{x^2 - 2\cos. \frac{2p\pi}{n} x + 1} dx.$$

and when these partial integrals are obtained, the entire ones will be found by putting $p = 0, 1 \ldots . \frac{n}{2}$ or $\frac{n-1}{2}$ according as p is even or odd.

Ex. 3. *Required the integral of*

$$\frac{x^r dx}{x^{2n} - 2ax^n + 1}$$

where a *is* < 1.

First let us find the quadratic factors of $x^{2n} - 2ax^n + 1$. For that purpose put

$$x^{2n} - 2ax^n = -1$$

Then

$$x^n = a \pm \sqrt{a^2 - 1}$$
$$= a \pm \sqrt{-1} . \sqrt{1 - a^2}$$

since a is < 1.

Now put $a = \cos. \delta$; then

$$x^n = \cos. \delta \pm \sqrt{-1} \sin. \delta$$
$$= \cos. (2p\pi + \delta) \pm \sqrt{-1} \sin. (2p\pi + \delta)$$
$$\therefore x = \cos. \frac{2p\pi + \delta}{n} \pm \sqrt{-1} \sin. \frac{2p\pi + \delta}{n}$$

and the general quadratic factor of

$$x^{2n} - 2ax^n + 1$$

is

$$x^2 - 2x \cos. \frac{2p\pi + \delta}{n} + 1$$

where p may be any number from 0, 1, &c. to $n - 1$.

Hence to find the general partial integral of the given indefinite difference, we assume

$$\frac{x^r}{x^{2n} - 2ax^n + 1} = \frac{P + Qx}{x^2 - 2\cos. \frac{2p\pi + \delta}{n} + 1} + \frac{N}{M}$$

and proceeding as in the last example, we get

$$Q = \sin. \frac{(r - 1 + 1)\ (2 p \pi + \delta)}{n} \times \frac{1}{n \sin. \delta}$$

and

$$P = \sin. \frac{(n - r) . (2 p \pi + \delta)}{n} \times \frac{1}{n \sin. \delta}$$

whence the remainder of the process is easy.

Case 4. *Let the factors of the denominator be all imaginary and equal in pairs.*

In this Case, we have the form

$$\frac{U}{V} = \frac{U}{\{(x+\alpha)^2 + \beta^2\}^n}$$

and assuming as in Case 2.

$$\frac{U}{V} = \frac{P + Q x}{(\overline{x + \alpha}|^2 + \beta^2)^n} + \frac{P' + Q' x}{(\overline{x + \alpha}|^2 + \beta^2)^{n-1}} + \&c.$$

$$+ \frac{K + L x}{(\overline{x + \alpha}|^2 + \beta^2)^2} + \frac{K' + L' x}{\overline{x + \alpha}|^2 + \beta^2}$$

and reducing to a common denominator,

$$U = P + Q x + (P' + Q' x)(\overline{x + \alpha}|^2 + \beta^2) + \&c.$$

and substituting for x one of its imaginary values, and equating homogeneous terms, in the result we get P and Q. Deriving from hence the values of $\frac{d U}{d x}$, $\frac{d^2 U}{d x^2}$, &c. and in each of these values substituting for x one of the quantities which makes $\overline{x + \alpha}|^2 + \beta^2 = 0$, and equating homogeneous terms we shall successively obtain

$$P', Q'; P'', Q'', \&c.$$

This method, however, not being very commodious in practice, for the present case, we shall recommend either the actual developement of the above expression according to the powers of x, and the comparison of the coefficients of the like powers (by art. 6), or the following method.

Having determined P and Q as above, make

$$U' = \frac{U - (P + Q x)}{\overline{x + \alpha}|^2 + \beta^2}$$

$$U'' = \frac{U' - (P' + Q' x)}{\overline{x + \alpha}|^2 + \beta^2}$$

$$U''' = \frac{U'' - (P'' + Q'' x)}{(x + \alpha)^2 + \beta^2}$$

$$\&c. = \&c.$$

Then since U′, U″, U‴, &c. have the same form as U, or have an

integer form, if we put for x that value which makes $(x+\alpha)^2+\beta^2=0$, and afterwards in the several results, equate homogeneous quantities we shall obtain the several coefficients.

$$P', Q'; \; P'', Q'', \&c.$$

Case 5. *If the denominator* V *consist of one set of Factors simple and unequal of the form*

$$x-\alpha \; x-\alpha', \&c.;$$

of several sets of equal simple Factors, as

$$(x-e)^p, (x-e')^q, \&c.$$

and of equal and unequal sets of quadratic factors of the forms

$$x^2+ax+b, \; x^2+a'x+b', \&c.$$

$$(x^2+lx+r)^\mu, (x^2+l'x+r')^\nu, \&c.$$

then the general assumption for obtaining the partial fractions must be

$$\frac{U}{V}=\frac{M}{x-\alpha}+\frac{M'}{x-\alpha'}+\&c.$$

$$+\frac{E}{(x-e)^p}+\frac{F}{(x-e)^{p-1}}+\&c. \frac{E'}{(x-e')^q}+\frac{F'}{(x-e')^{q-1}}+\&c.$$

$$+\frac{P+Qx}{x^2+ax+b}+\frac{P'+Q'x}{x^2+a'x+b'}+\&c.$$

$$+\frac{R+Sx}{(x^2+lx+r)^\mu}+\frac{R'+S'x}{(x^2+lx+r)^{\mu-1}}+\&c. \frac{G+Hx}{(x^2+l'x+r')^\nu}+\frac{G'+H'x}{(x^2+l'x+r')^{\nu-1}}+ \&c.$$

and the several coefficients may be found by applying the foregoing rules for each corresponding set. They may also be had at once by reducing to a common denominator both sides of the equation, and arranging the numerators according to the powers of x, and then equating homogeneous quantities.

We have thus shown that every rational fraction, whose denominator can be decomposed into simple or quadratic factors, may be itself analyzed into as many partial fractions as there are factors, and hence it is clear that the integral of the general function

$$\frac{Ax^m+Bx^{m-1}+\&c. \; Kx+L}{ax^n+bx^{n-1}+\&c. \; kx+l}dx$$

may, under these restrictions, always be obtained. It is always reducible, in short, to one or other or a combination of the forms

$$\int x^m dx, \int\frac{dx}{x+1}, \int\frac{dx}{x^2+1}.$$

Having disposed of rational forms we next consider irrational ones. Already (see Art. 86, &c.)

$$\int\frac{\pm dx}{\sqrt{(a-bx^2)}}, \int\frac{dx}{x\sqrt{(bx^2-a)}}, \int\frac{dx}{\sqrt{(ax-bx^2)}}$$

have been found in terms of circular arcs. We now proceed to treat of Irrationals generally; and the most natural and obvious way of so doing is to investigate such forms as admit of being *rationalized.*

90. *Required the integral of*

$$d\,x \times F\left\{x,\ x^{\frac{1}{m}},\ x^{\frac{1}{n}},\ x^{\frac{1}{p}},\ x^{\frac{1}{q}},\ \&c.\right\}$$

where F *denotes any rational function of the quantities between the brackets.*

Let

$$x = u^{m\,n\,p\,q}, \ \&c.$$

Then

$$x^{\frac{1}{m}} = u^{n\,p\,q\,r} \dots$$

$$x^{\frac{1}{n}} = u^{m\,p\,q\,r} \dots$$

$$x^{\frac{1}{p}} = u^{m\,n\,q\,r} \dots$$

$$\&c. = \&c.$$

and

$$d\,x = m\,n\,p\,q \dots \times u^{m\,n\,p\,q \dots - 1} \times d\,u$$

and substituting for these quantities in the above expression, it becomes rational, and consequently integrable by the preceding article.

Ex. $\dfrac{x^3 + 2\,a\,x^{\frac{2}{5}} + x^{\frac{1}{3}}}{b + c\,x^{\frac{1}{4}}}\,d\,x$

Here

$$x = u^{60}$$

$$x^3 = u^{180}$$

$$x^{\frac{2}{5}} = u^{24}$$

$$x^{\frac{1}{3}} = u^{20}$$

$$x^{\frac{1}{4}} = u^{15}$$

and

$$d\,x = 6\,u^{59}\,d\,u.$$

Hence the expression is transformed to

$$60\,u^{79}\,d\,u \cdot \frac{u^{160} + 2\,a\,u^4 + 1}{b + c\,u^{15}}$$

whose integral may be found by Art. 89, Case 3, Ex. 2.

91. *Required the integral of*

$$d\,x \times F\left\{x,\ (a + b\,x)^{\frac{1}{n}},\ (a + b\,x)^{\frac{1}{m}},\ \&c.\right\}$$

where F, as before, means any rational function.

Put $a + b\,x = u^{n\,m\,p \dots}$ then substitute, and we get

$$\frac{n\,m\,p \dots}{b} \cdot u^{n\,m\,p \dots - 1}\,d\,u \times F\left(\frac{u^{n\,m \dots} - a}{b},\ u^{m\,p \dots},\ u^{n\,p \dots},\ \&c.\right)$$

which is rational.

Examples to this general result are

$$\frac{x^4 d x}{c x^5 + (a + b x)^{\frac{3}{4}}} \text{ and } \frac{x^2 d x (a + b x)^{\frac{2}{3}}}{x + c (a + b x)^{\frac{2}{7}}},$$

which are easily resolved.

92. *Required the integral of*

$$d x F \left\{ x, \left(\frac{a + b x}{f + g x}\right)^{\frac{m}{n}}, \left(\frac{a + b x}{f + g x}\right)^{\frac{p}{q}}, \text{ \&c.} \right\}.$$

Assume

$$\frac{a + b x}{f + g x} = u^{n q s \cdots\cdots}$$

and then by substituting, the expression becomes rational and integrable.

93. *Required the integral of*

$$d x F \{x, \sqrt{} (a + b x + c x^2)\}$$

Case 1. When c is positive, let

$$a + b x + c x^2 = c (x + u)^2.$$

Then

$$x = \frac{a - c u^2}{2 c u - b} \text{ and } d x = - \frac{2 c (c u^2 - b u + a) d u}{(2 c u - b)^2}$$

$$\sqrt{} (a + b x + c x^2) = \frac{c u^2 - b u + a}{2 c u - b} \cdot \sqrt{} c$$

and substituting, the expression becomes *rational.*

Case 2. When c is negative, if r, r′ be the roots of the equation

$$a + b x - c x^2 = 0$$

Then assume

$$\sqrt{} c (x - r) (r' - x) = (x - r) c u$$

and we have

$$x = \frac{c r u^2 + r'}{c u^2 + 1}, \; d x = \frac{(r - r') 2 c u \, d u}{(c u^2 + 1)^2}$$

$$\sqrt{} (a + b x - c x^2) = \frac{(r' - r) c y}{c y^2 + 1}$$

and by substitution, the expression becomes rational.

94. *Required the integral of*

$$d x F \{x, (a + b x)^{\frac{1}{2}}, (a' + b' x)^{\frac{1}{2}}\}.$$

Make

$$a + b x = (a' + b' x) u^2;$$

Then

$$x = \frac{a - a' u^2}{b' u^2 - b}, \; d x = \frac{(a' b - b' a) 2 u \, d u}{(b' u^2 - b)^2}$$

$$\sqrt{} (a + b x) = \frac{u \sqrt{} (a b' - a' b)}{\sqrt{} (b' u^2 - b)}, \;\; \sqrt{} (a' + b' x) = \frac{\sqrt{} (a b' - a' b)}{\sqrt{} (b' u^2 - b)}.$$

Hence, substituting, the above expression becomes of the form

$$d u F' \{u, \sqrt{(b' u^2 - b)}\}$$

F' denoting a rational function different from that represented by F. But this form may be rationalized by 93; whence the expression becomes integrable.

95. *Required the integral of*

$$x^{m-1} d x (a + b x^n)^{\frac{p}{q}}.$$

This form may be rationalized when either $\frac{m}{n}$, or $\frac{m}{n} + \frac{p}{q}$ is an integer.

Case 1. Let $a + b x^n = u^q$; then $(a + b x^n)^{\frac{p}{q}} = u^p$, $x^n = \frac{u^q - a}{b}$, $x^m = \left(\frac{u^q - a}{b}\right)^{\frac{m}{n}}$, $x^{m-1} d x = \frac{q u^{q-1} d u}{n b} \left(\frac{u^q - a}{b}\right)^{\frac{m-n}{n}}$.

Hence the expression becomes

$$\frac{q}{n b} \cdot u^{p+q-1} d u \left(\frac{u^q - a}{b}\right)^{\frac{m-n}{n}}$$

which is rational and integrable when $\frac{m}{n}$ is an integer.

Case 2. Let $a + b x^n = x^n u^q$; then substituting as before, we get the transformed expression

$$\frac{q a^{\frac{m}{n} + \frac{p}{q}}}{n} \cdot \frac{u^{p+q-1} d u}{(u^q - b)^{\frac{m}{n} + \frac{p}{q} + 1}}$$

which is rational and integrable when $\frac{m}{n} + \frac{p}{q}$ is an integer.

Examples are

$$\frac{x^2 d x}{(a^2 + x^2)^{\frac{1}{2}}}, \quad \frac{x^{\pm 2 m} d x}{(a^2 + x^2)^{\frac{1}{2}}},$$

$$x^{-2 m} d x (a^2 + x^2)^{\frac{2 m + 1}{2}}, \quad \frac{x^6 d x}{(a^3 + x^3)^{\frac{1}{3}}}.$$

96. *Required the integral of*

$$x^{m-1} d x (a + b x^n)^{\frac{p}{q}} \times F(x^n).$$

This expression becomes rational in the same cases, and by the same substitutions, as that of 95. To this form belongs

$$x^{m+n-1} d x (a + b x^n)^{\frac{p}{q}}$$

and the more general one

$$\frac{P}{Q} x^{m-1} d x \times (a + b x)^{\frac{p}{q}}$$

where

$$P = A + B x^n + C x^{2n} + \&c.$$

and

$$Q = A' + B' x^n + C' x^{2n} + \&c.$$

97. *Required the integral of*

$$x^{m-1} d x \times F\{x^m, x^n, (a + b x^n)^{\frac{1}{q}}\}$$

Make $a + b x^n = u^q$; then

$$x^{m-1} d x = \frac{q u^{q-1} d u}{n b} . \left(\frac{u^q - a}{b}\right)^{\frac{m}{n}-1} d u$$

and in the cases where $\frac{m}{n}$ is an integer, the whole expression becomes rational and integrable.

98. *Required the integral of*

$$\frac{X d x}{X' + X'' + \sqrt{(a + b x + c x^2)}}$$

where X, X′, X″ *denote any rational functions of* x.

Multiply and divide by

$$X' + X'' - \sqrt{(a + b x + x^2)}$$

and the result is, after reduction,

$$\frac{X X' d x}{X'^2 - X''^2 (a + b x + c x^2)} - \frac{X X'' d x \sqrt{(a + b x + c x^2)}}{X'^2 - X''^2 (a + b x + c x^2)}$$

consisting of a rational and an irrational part. The irrational part, in many cases, may also be rationalized, and thus the whole made integrable.

99. *Required the integral of*

$$x^m d x F\{x^n, \sqrt{(a + b x^n + c x^{2n})}\}$$

Let $x^n = u$; then the expression may be transformed into

$$\frac{1}{n} u^{\frac{m+1}{n}-1} d u F\{u, \sqrt{(a + b u + c u^2)}\}$$

which may be rationalized by Art. 98, when $\frac{m+1}{n}$ is an integer.

100. *Required the integral of*

$$x^m d x F\{x^n, \sqrt{(a + b^2 x^{2n})}, b x^n + \sqrt{(a + b^2 x^{2n})}\}.$$

Let

$$b x^n + \sqrt{(a + b^2 x^{2n})} = u;$$

then

$$x^m d x = \frac{1}{n (2 b)^{\frac{m+1}{n}}} . \frac{u^2 + a}{u} . \left(\frac{u^2 - a}{u}\right)^{\frac{m+1}{n}-1} d u$$

and the whole expression evidently becomes rational when $\frac{m+1}{n}$ is an integer.

Many other general expressions may be rationalized, and much might

be said further upon the subject; but the foregoing cases will exhibit the general method of such reductions. If the reader be not satisfied let him consult a paper in the Philosophical Transactions for 1816, by E. Ffrench Bromhead, Esq. which is decidedly the best production upon the Integrals of Irrational Functions, which has ever appeared.

Perfect as is the theory of Rational Functions, yet the like has not been attained with regard to Irrational Functions. The above and similar artifices will lead to the integration of a vast number of forms, and to that of many which really occur in the resolution of philosophical and other problems; but a method universally applicable has not yet been discovered, and probably never will be.

Hitherto the integrals of algebraic forms have been investigated. We now proceed to Transcendental Functions.

101. *Required the integral of*

$$a^x dx.$$

By Art. 17,

$$d.a^x = l.a \times a^x dx$$

$$\therefore \int a^x dx = \frac{1}{la}\int da^x$$

$$= \frac{1}{la}.a^x + C \quad \ldots\ldots \quad (a)$$

Hence

$$\int a^{mx} dx = \frac{1}{m\,la} a^{mx} + C \quad \ldots\ldots \quad (b)$$

102. *Required the integral of*

$$X a^x dx$$

where X *is an algebraic function of* x.

By the form (see 73)

$$d(uv) = u\,dv + v\,du$$

we have

$$\int u\,dv = uv - \int v\,du.$$

Hence

$$\int X a^x dx = X.\frac{a^x}{la} - \int \frac{a^x}{la} dX$$

$$\int \frac{dX}{dx}.\frac{a^x dx}{la} = \frac{dX}{dx}.\frac{a^x}{(la)^2} - \int \frac{a^x}{(la)^2}\,\frac{d^2X}{dx}$$

$$\int \frac{d^2X}{dx^2}.\frac{a^x dx}{(la)^2} = \frac{d^2X}{dx^2}.\frac{a^x}{(la)^3} - \int \frac{a^x}{(la)^3}\,\frac{d^3X}{dx^2}$$

&c. = &c.

the law of continuation being manifest.

Hence, by substitution,

$$\int X a^x dx = X\frac{a^x}{l a} - \frac{dX}{dx}\cdot\frac{a^x}{(l a)^2} + \frac{d^2X}{dx^2}\cdot\frac{a^x}{(l a)^3} - \&c.$$

which will terminate when X is of the form

$$A + Bx + Cx^2 + \&c.$$

Ex. $\int x^3 a^x dx = \frac{a^x x^3}{l a} - \frac{3 a^x x^2}{(l a)^2} + \frac{3.2 a^x x}{(l a)^3} - \frac{3.2 a^x}{(l a)^4} + C.$

OTHERWISE

$$\int a^x X dx = a^x \int X dx - \int l a . a^x dx \int X dx$$
$$= a^x X' - l a \int a^x X' dx$$

putting

$$X' = \int X dx.$$

Hence

$$\int a^x X' dx = a^x X'' - l a \int a^x X'' dx$$
$$\&c. = \&c.$$

and substituting, we get

$$\int a^x . X dx = a^x X' - l a . a^x X'' + (l a)^2 a^x X''' - \&c.$$

X', X'', X''', &c. being equal to $\int X dx$, $\int X' dx$, $\int X'' dx$, &c. respectively.

Ex. $\int a^x \frac{dx}{x} = a^x l x + \frac{x l a}{1} + \frac{x^2 (l a)^2}{1.2.2} + \frac{x^3 (l a)^3}{1.2.3.3} + \&c. + C.$

which does not terminate.

By this last example we see how an Indefinite Difference may be integrated in an infinite series. If in that example x be supposed less than 1, the terms of the integral become less and less or the series is convergent. Hence then by taking a few of the first terms we get an approximate value of the integral, which in the absence of an exact one, will frequently suffice in practice.

The general formula for obtaining the integral in an infinite or finite series, corresponding to that of Taylor in the Calculus of Indefinite Differences, is the following one, ascribed to John Bernoulli, and usually termed

JOHN BERNOULLI'S THEOREM.

$$\int X dx = X x - \int x dX$$
$$\int \frac{dX}{dx} . x dx = \frac{dX}{dx}\cdot\frac{x^2}{2} - \int \frac{x^2 dx}{2}\cdot\frac{d^2X}{dx^2}$$
$$\int \frac{d^2X}{dx^2}\cdot\frac{x^2 dx}{2} = \frac{d^2X}{dx^2}\cdot\frac{x^3}{2.3} - \int \frac{x^3 dx}{2.3}\cdot\frac{d^3X}{dx^3}$$
$$\&c. = \&c.$$

Hence

$$\int X dx = Xx - \frac{dX}{dx} \cdot \frac{x^2}{2} + \frac{d^2X}{dx^2} \cdot \frac{x^3}{2.3} - \&c. + C$$

the theorem in question.

Ex. 1. $\int x^m dx = x^{m+1} - \frac{m}{2} x^{m+1} + \frac{m.m-1}{2.3} x^{m+1} - \&c. + C$

$$= x^{m+1} \times \left(1 - \frac{m}{2} + \frac{m.m-1}{2.3} + \&c.\right) + C$$

$$= \frac{x^{m+1}}{m+1}\left(m+1 - \frac{\overline{m+1}.m}{2} + \frac{\overline{m+1}.m.m-1}{2.3} - \&c.\right) + C$$

But since

$$(1-1)^{m+1} = 1 - \overline{m+1} + \frac{\overline{m+1}.m}{2} - \&c. = 0$$

$$\therefore \int x^m dx = \frac{x^{m+1}}{m+1} + C$$

as in Art. 78.

102. *Required the integral of*

$$X dx (lx)^n$$

where X *is any Algebraic Function of* x, lx *the Hyperbolic logarithm of* x, *and* n *a positive integer.*

By the formula

$$\int u dv = uv - \int v du$$

we have

$$\int X dx (lx)^n = (lx)^n \int X dx - n \int (lx)^{n-1} \frac{dx}{x} \int X dx$$

$$= (lx)^n X' - n \int (lx)^{n-1} \frac{dx}{x} X'$$

$$\int \frac{X'}{x} dx (lx)^{n-1} = (lx)^{n-1} X'' - (n-1) \int (lx)^{n-2} \frac{dx}{x} X''$$

$$\&c. = \&c.$$

where X', X'', X''', &c. are put for $\int X dx$, $\int \frac{X'}{x} dx$, $\int \frac{X''}{x} dx$, &c. respectively.

Hence

$$\int X dx (lx)^n = X'(lx)^n - nX''(lx)^{n-1} + n.(n-1)X'''(lx)^{n-2} - \&c. + C.$$

Ex. 1. $\int x^m dx (lx)^n = \frac{x^{m+1}}{m+1}\left\{(lx)^n - \frac{n}{m+1}(lx)^{n-1} \&c.\right\}$

193. *Required the integral of*

$$\frac{dx}{x}.U$$

where U *is any function of* l x.

Let $$u = l\,x.$$
Then
$$d\,u = \frac{d\,x}{x};$$
and substituting, the expression becomes algebraic, and therefore integrable in many cases.

104. *Required the integral of*
$$X\,d\,x\,(l\,x)^n$$
where n *is negative.*

Integrating *by Parts*, as it is termed, or by the formula
$$\int u\,d\,v = u\,v - \int v\,d\,u$$
we get, since
$$\int \frac{X\,d\,x}{(l\,x)^n} = \int X\,x\,.\,\frac{d\,x}{x}\,(l\,x)^{-n},$$
$$\int \frac{X\,d\,x}{(l\,x)^n} = -\frac{X\,x}{(n-1)\,(l\,x)^{n-1}} + \frac{1}{n-1}\,.\int \frac{d\,x}{(l\,x)^{n-1}}\,.\,\frac{d\,(X\,x)}{d\,x}$$
and pursuing the method, and writing
$$X' = \frac{d\,(X\,x)}{d\,x}$$
$$X'' = \frac{d\,(X'\,x)}{d\,x}$$
$$\&c. = \&c.$$
we have
$$\int \frac{X d x}{(l\,x)^n} = -\frac{X\,x}{(n-1)(l x)^{n-1}} - \frac{X'x}{n-1\,.\,n-2\,.(l x)^{n-2}} - \&c. - \int \frac{X^{(n)'}\,d\,x}{(n-1)\ldots 2\,.\,1\,(l x)}$$
or
$$-\frac{X\,x}{(n-1)\,(l\,x)^{n-1}} - \&c. - \int \frac{X^{(m+1)'}\,d\,x}{(n-1)\,.\,(n-2)\ldots\ldots(n-m)\,(l\,x)^{n-m}}$$
according as n is or is not an integer, m being in the latter case the greatest integer in n.
$$\text{Ex.} \int \frac{x^m\,d\,x}{(l\,x)^n} = -\frac{x^{m+1}}{n-1}\left\{\frac{1}{(l\,x)^{n-1}} + \frac{m+1}{(n-2)\,(l\,x)^{n-2}} + \&c.\right\}$$
$$-\frac{(m+1)^{n-1}}{(n-1)\,(n-2)\ldots\ldots 1}\int \frac{x^m\,d\,x}{l\,x}$$
when m is an integer.

105. *Required the integrals of*
$$d\,\theta\,.\,\cos.\,\theta,\ d\,\theta\,.\,\sin.\,\theta,\ d\,\theta\,.\,\tan.\,\theta,\ d\,\theta\,.\,\sec.\,\theta,\ \frac{d\,\theta}{\cos.\,\theta},\ \frac{d\,\theta}{\sin.\,\theta},\ \frac{d\,\theta}{\tan.\,\theta}.$$
By Art. 26, &c.
$$d\,\sin.\,\theta = d\,\theta\,.\,\cos.\,\theta, \text{ and } d\,\cos.\,\theta = -\,d\,\theta\,\sin.\,\theta$$
$$\therefore \int d\,\theta\,\cos.\,\theta = \sin.\,\theta + C \quad \ldots\ldots\ldots\ldots \quad (a)$$
and
$$\int d\,\theta\,\sin.\,\theta = C - \cos.\,\theta \quad \ldots\ldots\ldots\ldots \quad (b)$$

Again let tan. $\theta = t$; then

$$d\theta = \frac{dt}{1+t^2}$$

and

$$\int d\theta \tan.\theta = \int \frac{t\,dt}{1+t^2} = \tfrac{1}{2}\,l\,(1+t^2) + C$$
$$= C - l.\cos.\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

since

$$1 + t^2 = \sec.^2\theta = \frac{1}{\cos.^2\theta}.$$

Again

$$d\theta \sec.\theta = \frac{d\theta}{\cos.\theta} = \frac{d\theta\cos.\theta}{1-\sin.^2\theta}$$
$$= \frac{d(\sin.\theta)}{1-\sin.^2\theta}$$
$$= \tfrac{1}{2}\cdot\frac{d(\sin.\theta)}{1-\sin.\theta} + \tfrac{1}{2}\cdot\frac{d\sin.\theta}{1+\sin.\theta}$$
$$\therefore \int d\theta\sec.\theta = \tfrac{1}{2}l.(1+\sin.\theta) - \tfrac{1}{2}l\,(1-\sin.\theta) + C$$
$$= l.\tan.\left(45^\circ + \frac{\theta}{2}\right) + C \; . \; . \; . \; (d)$$

which is the same as $\int \frac{d\theta}{\cos.\theta}$.

Again

$$\int \frac{d\theta}{\sin.\theta} = \int d\theta\,\mathrm{cosec.}\,\theta$$
$$= \int d\theta\sec.\left(\frac{\pi}{2}-\theta\right) = -\int d.\left(\frac{\pi}{2}-\theta\right)\sec.\left(\frac{\pi}{2}-\theta\right)$$
$$= -l.\tan.\left(45^\circ + \frac{\pi}{4} - \frac{\theta}{2}\right) + C$$
$$= l.\left(\tan.\frac{\theta}{2}\right) + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Again

$$\int \frac{d\theta}{\tan.\theta} = \int d\theta\cot.\theta = \int d\theta.\tan.\left(\frac{\pi}{2}-\theta\right)$$
$$= l\cos.\left(\frac{\pi}{2}-\theta\right) + C \text{ (by c)}$$
$$= l.\sin.\theta + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

106. *Required the integral of*

$$\sin.^m\theta\cos.^n\theta.\,d\theta.$$

m *and* n *being positive or negative integers.*

Let sin. $\theta = u$; then $d\,\theta \cos.\theta = d\,u$ and the above expression becomes

$$u^m\, d\,u\,(1-u^2)^{\frac{n-1}{2}}$$

which is integrable when either $\frac{m+1}{2}$ or $\frac{m+1}{2} + \frac{n-1}{2} = \frac{m+n}{2}$ is an integer (see 95.) If n be odd, the radical disappears; if n be even and m even also, then $\frac{m+n}{2}$ = an integer; if n be even and m odd, then $\frac{m+1}{2}$ is an integer. Whence

$$u^m\, d\,u\,(1-u^2)^{\frac{n-1}{2}}$$

is integrable by 95.

OTHERWISE,

Integrating by Parts, we have

$$\int d\,\theta \sin.^m\theta \cos.^n\theta = -\frac{\sin.^{m-1}\theta}{n+1}\cos.^{n+1}\theta + \frac{m-1}{m+1}\int \cos.^{n+2}\theta.\sin.^{m-2}\theta \times d\,\theta$$

$$= -\frac{\sin.^{m-1}\theta}{m+n}\cos.^{n+1}\theta + \frac{m-1}{m+n}\int d\,x \sin.^{m-2}\theta \cos.^n\theta$$

and continuing the process m is diminished by 2 each time.

In the same way we find

$$\int d\,\theta \sin.^m\theta \cos.^n\theta = \frac{\sin.^{m+1}\theta \cos.^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int d\,x \sin.^m\theta \cos.^{n-2}\theta$$

and so on.

107. *Required the integrals of*

$$d\,u = d\,\theta \sin.(a\,\theta + b)\cos.(a'\,\theta + b')$$

$$d\,v = d\,\theta \sin.(a\,\theta + b)\sin.(a'\,\theta + b')$$

and

$$d\,w = d\,\theta \cos.(a\,\theta + b)\cos.(a'\,\theta + b')$$

By the known forms of Trigonometry we have

$$d\,u = d\,\theta\,\{\sin.(\overline{a+a'}.\theta + b + b') + \sin.(\overline{a-a'}.\theta + b - b')\}$$

$$d\,v = d\,\theta\,\{\cos.(\overline{a+a'}.\theta + b + b') - \cos.(\overline{a-a'}.\theta + b - b')\}$$

$$d\,w = d\,\theta\,\{\cos.(\overline{a+a'}.\theta + b + b') + \cos.(\overline{a-a'}.\theta + b - b')\}$$

Hence by 105 we have

$$u = C - \tfrac{1}{2}\left\{\frac{\cos.(\overline{a+a'}.\theta + b + b')}{a+a'} + \frac{\cos.(\overline{a-a'}.\theta + b - b')}{a-a'}\right\}$$

$$v = C + \tfrac{1}{2}\left\{\frac{\sin.(\overline{a+a'}.\theta + b + b')}{a+a'} - \frac{\sin.(\overline{a-a'}.\theta + b - b')}{a-a'}\right\}$$

$$w = C + \tfrac{1}{2}\left\{\frac{\sin.(\overline{a+a'}.\theta + b + b')}{a+a'} + \frac{\sin.(\overline{a-a'}.\theta + b - b')}{a-a'}\right\}$$

These integrals are very useful.

108. *Required the integrals of*

$$\theta^n\, d\,\theta \sin.\,\theta, \text{ and } \theta^n\, d\,\theta \cos.\,\theta.$$

Integrating by Parts we get

$\int \theta^n \times d\,\theta \sin.\,\theta = C - \theta^n \cos.\,\theta + n\,\theta^{n-1} \sin.\,\theta + n\,.\,(n-1)\,\theta^{n-2} \cos.\,\theta - \&c.$

and

$\int \theta^n \times d\,\theta \cos.\,\theta = C + \theta^n \sin.\,\theta + n\,\theta^{n-1} \cos.\,\theta - n\,.\,(n-1)\,\theta^{n-2} \sin.\,\theta + \&c.$

109. *Required the integrals of*

$$\begin{gathered} X\,d\,x \sin.^{-1} x \\ X\,d\,x \tan.^{-1} x \\ X\,d\,x \sec.^{-1} x \\ \&c. \end{gathered}$$

Integrating by Parts we have

$$\int X\,d\,x \sin.^{-1} x = \sin.^{-1} x \int X\,d\,x - \int \frac{d\,x \int X\,d\,x}{\sqrt{(1-x^2)}}$$

$$\int X\,d\,x \tan.^{-1} x = \tan.^{-1} x \int X\,d\,x - \int \frac{d\,x \int X\,d\,x}{1+x^2}$$

$$\int X\,d\,x \sec.^{-1} x = \sec.^{-1} x \int X\,d\,x - \int \frac{d\,x \int X\,d\,x}{x\sqrt{(x^2-1)}}$$

$$\&c. = \&c.$$

see Art. 86.

110. *Required the integral of*

$$d\,u = \frac{(f + g \cos.\,\theta)\,d\,\theta}{(a + b \cos.\,\theta)^n}.$$

Integrating by Parts and reducing, we have

$$u = \frac{(a\,g - b\,f) \sin.\,\theta}{(n-1)(a^2-b^2)(a + b\cos.\,\theta)^{n-1}} + \frac{1}{(n-1)(a^2-b^2)} \times$$

$$\int \frac{(n-1)(a\,f - b\,g) + (n-2)(a\,g - b\,f)\cos.\,\theta}{(a + b\cos.\,\theta)^{n-1}}\,d\,\theta,$$

which repeated, will finally produce, when n is an integer, the integral required.

$$\text{Ex. } \int \frac{d\,\theta}{a + b\cos.\,\theta} = \frac{2}{\sqrt{(a^2-b^2)}}\,.\,\tan.^{-1} \frac{(a-b)\tan.\,\frac{\theta}{2}}{\sqrt{(a^2-b^2)}} + C$$

or

$$\frac{1}{\sqrt{(b^2-a^2)}}\,.\,l.\,\frac{b + a\cos.\,\theta + \sin.\,\theta\,\sqrt{(b^2-a^2)}}{a + b\cos.\,\theta} + C.$$

Notwithstanding the numerous forms which are integrable by the preceding methods, there are innumerable others which have hitherto resisted all the ingenuity that has been employed to resolve them. If any such appear in the resolution of problems, they must be expanded into con-

verging series, by some such method as that already delivered in Art. 101; or with greater certainty of attaining the requisite degree of convergency, by the following

METHOD OF APPROXIMATION.

111. *Required to integrate between* x = b, x = a, *any given Indefinite Difference, in a convergent series.*

Let f (x) denote the exact integral of $\int X\, d\, x$; then by Taylor's Theorem

$$f.(x+h) - fx = Xh + \frac{dX}{dx}\frac{h^2}{1.2} + \&c.$$

and making

$$h = b - a$$

$$f(x + b - a) - fx = X.(b-a) + \frac{dX}{dx}.\frac{(b-a)^2}{1.2} + \&c.$$

Again, make

$$x = a$$

then

$$\frac{dX}{dx}, \frac{d^2X}{dx^2}, \&c.$$

become constants

$$A, A', \&c.$$

and we obtain

$$f(b) - f(a) = A(b-a) + \frac{A'}{2}.(b-a)^2 + \frac{A''}{2.3}(b-a)^3$$

which, when b — a is small compared with unity, is sufficiently convergent for all practical purposes.

If b — a be not small, assume

$$b - a = p.\beta$$

p being the number of equal parts β, into which the interval b — a is supposed to be divided, in order to make β small compared with unity. Then taking the integral between the several limits

$$a,\ a+\beta$$
$$a,\ a+2\beta$$
$$\&c.$$
$$a,\ a+p\beta$$

we get

$$f.(a+\beta)-f(a)=A\beta+\frac{A'}{2}.\beta^2+\frac{A''}{2.3}.\beta^3+\&c.$$

$$f(a+2\beta)-f(a+\beta)=B\beta+\frac{B'}{2}.\beta^2+\frac{B''}{2.3}\beta^3+\&c.$$

$$\&c. = \&c.$$

$$f(a+p\beta)-f(a+\overline{p-1}.\beta)=P\beta+\frac{P'}{2}\beta^2+\frac{P''}{2.3}\beta^3+\&c.$$

A, A′, &c. B, B′, &c. . . . . . . P, P′, &c.

being the values of

$$X, \frac{dX}{dx}, \&c.$$

when for x we put

$$a, a+\beta, a+2\beta, \&c.$$

Hence

$$\begin{aligned} f(b)-f(a) &= (A+B+\ldots\ldots P)\beta \\ &+ (A'+B'+\ldots\ldots P')\frac{\beta^2}{1.2} \\ &+ (A''+B''+\ldots\ldots P'')\frac{\beta^3}{1.2.3} \\ &+ \&c. \end{aligned}$$

the integral required, the convergency of the series being of any degree that may be demanded.

If β be taken very small, then

$$f(b)-f(a)=(A+B+\ldots\ldots P)\beta \text{ nearly.}$$

Ex. *Required the approximate value of*

$$\int X^{m-1}dx\times(1-x^n)^{\frac{p}{q}}$$

between the limits of $x=0$ and $x=1$, *when neither* $\frac{m}{n}$, *nor* $\frac{m}{n}+\frac{p}{q}$ *is an integer.*

Here

$$X=x^{m-1}(1-x^n)^{\frac{p}{q}}$$

and

$$\frac{dX}{dx}=(m+n\frac{p}{q}-1)x^{m-2}(1-x^n)^{\frac{p}{q}}-\frac{np}{q}x^{m-2}(1-x^n)^{\frac{p}{q}-1}$$

$b-a=1-0=1.$

Assume $1=10\times\beta$, and we have for limits

$$0, \frac{1}{10};\ 0, \frac{2}{10};\ \&c.$$

Hence m being > 1,

$$A = 0$$

$$B = \frac{1}{10^{m-1}}\left(1 - \frac{1}{10^n}\right)^{\frac{p}{q}}$$

$$C = \left(\frac{2}{10}\right)^{m-1}\left\{1 - \left(\frac{2}{10}\right)^n\right\}^{\frac{p}{q}}$$

$$D = \left(\frac{3}{10}\right)^{m-1}\left\{1 - \left(\frac{3}{10}\right)^n\right\}^{\frac{p}{q}}$$

$$\&c. = \&c.$$

$$P = \left(\frac{9}{10}\right)^{m-1}\left\{1 - \left(\frac{9}{10}\right)^n\right\}^{\frac{p}{q}}.$$

Hence, between the limits $x = 1$ and $x = 0$

$$\int X\,d\,x = \frac{1}{10^{m+n\frac{p}{q}}} \times \left\{(10^n - 1)^{\frac{p}{q}} + (10^n - 2^n)^{\frac{p}{q}}\right.$$

$$\left. + (10^n - 3^n)^{\frac{p}{q}} + \&c. + (10^n - 9^n)^{\frac{p}{q}}\right\} \text{ nearly.}$$

We shall meet with more particular instances in the course of our comments upon the text.

Hitherto the use of the Integral Calculus of Indefinite Differences has not been very apparent. We have contented ourselves so far with making as rapid a sketch as possible of the leading principles on which the Inverse Method depends; but we now come to its

APPLICATIONS.

112. *Required to find the area of any curve, comprised between two given values of its ordinate.*

Let E c C (fig. to LEMMA II of the text) be a given or definite area comprised between 0 and C c, or 0 and y. Then C c being fixed or Definite, let B b be considered Indefinite, or let L b = d y. Hence the Indefinite Difference of the area E c C is the Indefinite area

B C c b.

Hence if E C = x, and S denote the area E c C; then

$$d\,S = B\,C\,c\,b = C\,L + L\,c\,b$$

$$= y\,d\,x + L\,c\,b.$$

But L c b is *heterogeneous* (see Art. 60) compared with C L or y d x.

$$\therefore d\,S = y\,d\,x$$

Hence

$$S = \int y\,dx,$$

the area required.

Ex. 1. *Required the area of the common parabola.*

Here

$$y^2 = a\,x.$$

$$\therefore d\,x = \frac{2\,y\,d\,y}{a}$$

and

$$S = \int \frac{2\,y^2\,d\,y}{a} = \frac{2}{3\,a}\,y^3 + C;$$

and between the limits of $y = r$ and $y = r'$ becomes

$$S = \frac{2}{3\,a}\,(r^3 - r'^3)$$

If m and m′ be the corresponding values of x, we have

$$S = \frac{2}{3}\,(r\,m - r'\,m')$$

$$= \frac{2}{3} \text{ of the circumscribing rectangle.}$$

Let $r' = 0$, then

$$S = \frac{2}{3}\,r\,m \text{ (see Art. 21.)}$$

Ex. 2. Take the general Parabola whose equation is

$$y^m = a\,x^n.$$

Here it will be found in like manner that

$$S = \frac{m\,x\,y}{m+n} + C$$

$$= \frac{m}{m+n}\cdot\alpha\,\beta$$

between the limits of $n = y = 0$, and $x = \alpha$, $y = \beta$.

Hence all Parabolas may be squared, as it is termed; or a square may be found whose area shall be equal to that of any Parabola.

Ex. 3. *Required the area of an Hyperbola comprised by its asymptote, and one infinite branch.*

If x, y be parallel to the asymptotes, and originate in the center

$$x\,y = a\,b$$

is the equation to the curve.

Hence

$$d\,x = -\frac{a\,b\,d\,y}{y^2}$$

and

$$S = \int - \frac{a\,b\,d\,y}{y} = C - a\,b\,l\,y.$$

Let at the vertex $y = \beta$, and $x = 0$; then the area is 0 and

$$C = a\,b\,.\,l\,\beta.$$

Hence

$$S = a\,b\,.\,l.\,\frac{\beta}{y}.$$

113. *If the curve be referred to a fixed center by the radius-vector* ϱ *and traced-angle* θ; *then*

$$d\,S = \frac{\varrho^2\,d\,\theta}{2}.$$

For $d\,S =$ the Indefinite Area contained by ϱ, and $\varrho + d\,\varrho = (\varrho + d\,\varrho)\,\frac{\varrho\,\sin.\,d\,\theta}{2}$ $= \frac{\varrho^2\,d\,\theta}{2} + \frac{\varrho\,d\,\varrho\,d\,\theta}{2}$ (Art. 26) and equating *homogeneous* quantities we have

$$d\,S = \frac{\varrho^2\,d\,\theta}{2}.$$

Ex. 1. In the Spiral of Archimedes

$$\varrho = a\,\theta$$

$$\therefore S = \frac{a^2}{2}\int \theta^2\,d\,\theta = \frac{a^2}{2.\,3}\,.\,\theta^3 + C.$$

Ex. 2. In the Trisectrix

$$\varrho = 2\cos.\,\theta \pm 1$$

$$\therefore d\,S = \tfrac{1}{2}\int (2\cos.\,\theta \pm 1)^2\,d\,\theta$$

which may easily be integrated.

Hence then the area of every curve could be found, if all integrations were possible. By such as are possible, and the general method of approximation (Art. 111) the quadrature of a curve may be effected either exactly or to any required degree of accuracy. In Section VII and many other parts of the Principia our author integrates Functions by means of curves; that is, he reduces them to areas, and takes it for granted that such areas can be investigated.

114. *To find the length of any curve comprised within given values of the ordinate; or To* RECTIFY *any curve.*

Let s be the length required. Then $d\,s =$ its Indefinite Chord, by Art. 25 and LEMMA VII.

$$\therefore d\,s = \sqrt{(d\,x^2 + d\,y^2)}$$

and

$$s = \int \sqrt{(d\,x^2 + d\,y^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Ex. 1. *In the general parabola*

$$y^m = a x^n.$$

Hence

$$d x^2 = \frac{m^2}{n^2 a^{\frac{2}{n}}} y^{\frac{2m}{n}-2} . d y^2$$

and

$$d s = d y . \sqrt{\left(1 + \frac{m^2}{n^2 a^{\frac{2}{n}}} y^{\frac{2m}{n}-2}\right)}$$

which is integrable by Art. 95 when either

$$\frac{1}{\frac{2m}{n} - 2} \text{ or } \frac{1}{\frac{2m}{n} - 2} + \frac{1}{2}$$

that is, when either

$$\frac{1}{2} . \frac{n}{m-n} \text{ or } \frac{1}{2} . \frac{m}{m-n}$$

is an integer; that is when either m or n is even.

The common parabola is Rectifiable, because then m = 2. In this case

$$d s = d y \sqrt{(1 + \frac{4}{a^2} y^2)} \quad . \quad . \quad . \quad . \quad . \quad (r)$$

Hence assuming according to Case 2 of Art. 95,

$$1 + \frac{4}{a^2} y^2 = y^2 u^2$$

we get the Rational Form

$$d s = \frac{u^2 d u}{\left(u^2 - \frac{4}{a^2}\right)^2}$$

Hence by Art. 89, Case 2,

$$s = \frac{u}{2\left(\frac{4}{a^2} - u^2\right)} - \frac{a}{8} l . \frac{\frac{2}{a} + \sqrt{u}}{\frac{2}{a} - \sqrt{u}} + C$$

$$= \frac{u}{2\left(\frac{4}{a^2} - u^2\right)} - \frac{a}{4} l . \frac{\frac{2}{a} + \sqrt{u}}{\sqrt{\frac{4}{a^2} - u}} + C.$$

But $u = \sqrt{\frac{1 + \frac{4}{a^2} y^2}{y^2}}$. Hence by substituting and making the necessary reductions

$$s = \frac{y\sqrt{\left(y^2+\frac{a^2}{4}\right)}}{a} + a\,l.\frac{y+\sqrt{\left(y^2+\frac{a^2}{4}\right)}}{\frac{a}{2}} + C.$$

Let $y = 0$; then $s = 0$ and we get $C = 0$
and $\therefore$ between the Limits of $y = 0$ and $y = \beta$

$$s = \frac{\beta.\sqrt{\left(\beta^2+\frac{a^2}{4}\right)}}{a} + a\,l.\frac{\beta+\sqrt{\left(\beta^2+\frac{a^2}{4}\right)}}{\frac{a}{2}}.$$

In the Second Cubical Parabola

$$y^3 = a\,x^2$$

and

$$d\,s = d\,y\sqrt{\left(1+\frac{9\,y}{4\,a}\right)}$$

which gives at once (Art. 91)

$$s = \frac{8}{27}a\left(1+\frac{9\,y}{4\,a}\right)^{\frac{3}{2}} + C.$$

Ex. 2. *In the circle* (Art. 26)

$$d\,s = \frac{d\,y}{\sqrt{(1-y^2)}}$$

which admits of Integration in a series only. Expanding $(1-y^2)^{-\frac{1}{2}}$ by the Binomial Theorem, we have

$$(1-y^2)^{-\frac{1}{2}} = 1 + \frac{y^2}{2} + \frac{1.3}{2.4}y^4 + \&c.$$

Hence,

$$d\,s = d\,y + \frac{y^2\,d\,y}{2} + \frac{1.3}{2.4}y^4\,d\,y + \&c.$$

and

$$s = y + \frac{y^3}{2.3} + \frac{3}{2.4.5}\cdot y^5 + \&c. + C$$

and between the limits of $y = 0$ and $y = \frac{1}{2}$ or for an arc of 30° we have

$$s = \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \&c.$$

$$= \frac{1}{2} + \frac{1}{3.2^4} + \frac{3}{5.2^8} + \frac{5}{7.2^{11}} + \frac{5.7}{9.2^{16}} + \&c.$$

$$= \left\{\begin{array}{l} .5 \\ .0208333333 \\ .0023437500 \\ .0003487720 \\ .0000593390 \end{array}\right\} = .5235851943 \text{ nearly.}$$

&c.

Hence 180° of the circle whose *radius* is 1 or the whole circumference π of the circle whose *diameter* is 1 is

$$\pi = .5235851943 \ldots \times 6 \text{ nearly}$$
$$= 3.1415111658$$

which is true to the fourth decimal place; or the defect is less than $\frac{1}{10000}$.
By taking more terms any required approximation to the value of π may be obtained.

Ex. 3. *In the Ellipse*

$$s = \int d x . \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}$$

where x is the abscissa referred to the center, a the semi-axis major and a e the eccentricity (see Solutions to Cambridge Problems, Vol. II. p. 144.)

115. *If the curve be referred to polar coordinates,* ϱ *and* θ; then

$$s = \int \sqrt{(\varrho^2 d \theta^2 + d \varrho^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

For

$$y = \varrho \sin. \theta$$
$$x = m + \varrho \cos. \theta$$

and if $d x^2$, $d y^2$ be thence found and substituted in the expression (114. a) the result will be as above.

Ex. 1. *In the Spiral of Archimedes*

$$\varrho = a \theta$$

$$\therefore d s = d \varrho \sqrt{\left(\frac{\varrho^2}{a^2} + 1\right)}$$

$$\therefore s = \frac{\varrho \sqrt{(\varrho^2 + a^2)}}{2 a} + 2 a l . \frac{\varrho + \sqrt{(\varrho^2 + a^2)}}{a} + C$$

see the value for s in the common parabola, Art. 114.

Ex. 2. In the logarithmic Spiral

$$\varrho = e^{\theta}$$

or

$$\theta = l . \varrho$$

and we find

$$s = \sqrt{2} \int d \varrho = \varrho \sqrt{2} + C.$$

116. *Required the Volume or solid Content of any solid formed by the revolution of a curve round its axis.*

Let V be the volume between the values 0 and y of the ordinate of the generating curve. Then d V = a cylinder whose base is πy^2 and altitude d x + a quantity *Indefinite* or *heterogeneous* compared with either d V or the cylinder.

But the cylinder $= \pi y^2 dx$. Hence equating *homogeneous* terms, we have

$$dV = \pi y^2 dx$$

and

$$V = \pi \int y^2 dx, \quad \ldots\ldots\ldots \quad (c)$$

Ex. 1. *In the sphere* (rad. $= r$)

$$y^2 = r^2 - x^2$$

$$\therefore V = \pi \int r^2 dx - \pi \int x^2 dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3}\right) + C;$$

and between the limits $x = 0$ and r

$$V = \frac{2}{3} r^3 \pi$$

which gives the Hemisphere.

Hence for the whole sphere

$$V = \frac{4}{3} r^3 \pi.$$

Ex. 2. *In the Paraboloid.*

$$y^2 = a x$$

$$\therefore V = \pi \int a x d x$$

$$= \frac{\pi a}{2} x^2 + C;$$

and between the limits $x = 0$ and α

$$V = \frac{\pi a}{2} . \alpha^2.$$

Ex. 3. *In the Ellipsoid.*

$$y^2 = \frac{b^2}{a^2} . (a^2 - x^2)$$

$$\therefore V = \frac{\pi b^2}{a^2} . \int (a^2 dx - x^2 dx)$$

$$= \frac{\pi b^2}{a^2} \left(a^2 x - \frac{x^3}{3}\right) + C;$$

and between the limits $x = 0$ and a

$$V = \frac{2 \pi b^2}{3 a^2} a^3 = \frac{2 \pi}{3} . a b^2.$$

Hence for the whole Ellipsoid

$$V = \frac{4}{3} \pi a b^2.$$

The formula (c) may be transformed to

$$V = \pi y S - \pi \int S dy \quad \ldots\ldots\ldots \quad (d)$$

where $S = \int y\,dx$ or the area of the generating curve, which is a singular expression, $\int S\,dy$ being also an area.

In philosophical inquiries solids of revolution are the only ones almost that we meet with. Thus the Sun, Planets and Secondaries are Ellipsoids of different eccentricities, or approximately such. Hence then in preparation for such inquiry it would not be of great use to investigate the Volumes of Bodies in general.

If x, y, z, denote the rectangular coordinates, or the perpendiculars let fall from any point of a *curved surface* upon three planes passing through a point given in position at right angles to one another, then it may easily be shown by the principles upon which we have all along proceeded, that

$$\left.\begin{aligned} dV &= dy\int z\,dx \\ \text{or}\quad &= dz\int y\,dx \\ \text{or}\quad &= dx\int z\,dy \end{aligned}\right\} \quad \ldots\ldots \quad (e)$$

according as we take the base of d V in the planes to which z, y, or x is respectively perpendicular

For let the Volume V be cut off by a plane passing through the point in the surface and parallel to any of the coordinate planes; then the area of the plane section thus made will be

$$\left.\begin{aligned} &\int z\,dx \\ \text{or}\quad &\int y\,dx \\ \text{or}\quad &\int z\,dy \end{aligned}\right\} \text{see Art. 112.}$$

Then another section, parallel to $\int z\,dx$, or $\int y\,dx$, or $\int z\,dy$ and at the Indefinite distance d y, or d z, or d x from the former being made, the Indefinite Difference of the Volume will be the portion comprised by these two sections; and the only thing then to be proved is that this portion is $= dy\int z\,dx$ or $dz\int y\,dx$, or $dx\int z\,dy$. But this is easily to be proved by LEMMA VII.

This, which is an easier and more comprehensible method of deducing d V than the one usually given by means of Taylor's Theorem, we have merely sketched; it being incompatible with our limits to enter into detail. To conclude we may remark that in Integrating both $\int z\,dx$, and $\int dy\int z\,dx$ must be taken within the prescribed limits, first considering *y Definite* and then *x*.

117. *To find the curved surface of a Solid of Revolution.*

Let the curved surface taken as far as the value y of the ordinate referred to the axis of revolution be σ, and s the length of the generating curve to that point; then $d\sigma =$ the surface of a cylinder the radius of whose base is y and circumference $2\pi y$, and altitude ds, by LEMMA VII. and like considerations. Hence

$$d\sigma = 2\pi y\,ds$$

and

$$\sigma = 2\pi\int y\,ds \quad \ldots \ldots \quad (a)$$

or

$$= 2\pi y s - 2\pi\int s\,dy \quad \ldots \ldots \quad (b)$$

which latter form may be used when s is known in terms of y; this will not often be the case however.

Ex. *In the common Paraboloid.*

$$y^2 = a x$$

and

$$\sigma = \frac{4\pi}{a}\int y\,dy\,\sqrt{(y^2 + a^2)}$$

$$= \frac{4\pi}{3a}(y^2 + a^2)^{\frac{3}{2}} + C.$$

Let $y = 0$ and β, then σ between these limits is expressed by

$$\sigma = \frac{4\pi}{3a}(\beta^2 + a^2)^{\frac{3}{2}}$$

If the surface of any solid whatever were required, by considerations similar to those by which (116. e) is established, we shall have

$$d\sigma = \sqrt{(dy^2 + dz^2)}\int\sqrt{(dx^2 + dz^2)} \quad \ldots \quad (c)$$

and substituting for dz in $\sqrt{\overline{dx^2 + dz^2}}$ its value deduced from $z = f.(x, y)$ on the supposition that y is Definite; and in $\sqrt{(dy^2 + dz^2)}$ its value supposing x Definite. Integrate first $\sqrt{(dx^2 + dz^2)}$ between the prescribed limits supposing y Definite and then Integrate $\sqrt{(dy^2 + dz^2)}\int\sqrt{(dx^2 + dz^2)}$ between its limits making x Definite. This last result will be the surface required.

We must now close our Introduction as it relates to the Integration of Functions of one *Independent* variable.

It remains for us to give a brief notice of the artifices by which Functions of two Independent Variables may be Integrated.

118. *Required the Integral of*

$$X\,dx + Y\,dy = 0,$$

where X *is any function of* x, *and* Y *a function of* y *the same or different.*

When each of the terms can be Integrated separately by the preceding methods for functions of *one* variable, the above form may be Integrated, and we have

$$\int X\, d x + \int Y\, d y = C.$$

This is so plain as to need no illustration from examples. We shall, nowever, give some to show how Integrals apparently Transcendental may in particular cases, be rendered algebraic.

Ex. 1. $\frac{d x}{x} + \frac{d y}{y} = 0.$

$$\therefore l\, x + l\, y = C = l.\, C'$$

$$\therefore l\,(x\, y) = l.\, C'$$

and

$$\therefore x\, y = C' \text{ or } = C.$$

Ex. 2. $\frac{d x}{\surd\,(1 - x^2)} + \frac{d y}{\surd\,(1 - y^2)} = 0.$

Here

$$\sin.^{-1} x + \sin.^{-1} y = C = \sin.^{-1} C$$

$$\begin{aligned}\therefore C &= \sin.\,\{\sin.^{-1} x = \sin.^{-1} y\} \\ &= \sin.\,(\sin.^{-1} x).\cos.\,(\sin.^{-1} y) + \cos.\,(\sin.^{-1} x) \sin.\,(\sin.^{-1} y) \\ &= x.\surd\,(1 - y^2) + y \surd\,(1 - x^2)\end{aligned}$$

which is algebraic.

Generally if the Integral be of the form

$$f^{-1}(x) + f.^{-1}(y) = C$$

Then assume

$$C = f.^{-1}(C)$$

and take the inverse function of $f^{-1}(C)$ and we have

$$C = f\,\{f^{-1}(x) + f^{-1}(y)\}$$

which when expanded will be algebraic.

119. *Required the Integral of*

$$Y\, d x + X\, d y = 0.$$

Dividing by X Y we get

$$\frac{d x}{X} + \frac{d y}{Y} = 0$$

which is Integrable by art. 118.

120. *Required the Integral of*

$$P\, d x + Q\, d y = 0;$$

where P *and* Q *are each such functions of* x *and* y *that the sum of the exponents of* x *and* y *in every term of the equation is the same.*

Let $x = u\,y$. Then if m be the constant sum of the exponents, P and Q will be of the forms

$$U \times y^m — U' y^m$$

U and U′ being functions of u.

Hence, since $d\,x = u\,d\,y + y\,d\,u$, we have

$$U . (u\,d\,y + y\,d\,u) + U'\,d\,y = 0$$

and

$$(U\,u + U')\,d\,y + U\,y\,d\,u = 0$$

$$\therefore \frac{d\,y}{y} + \frac{U\,d\,u}{U\,u + U'} = 0 \quad . \; . \; . \; . \; . \; . \quad (a)$$

which is Integrable by art. 118.

Ex. 1. $(a\,x + b\,y)\,d\,y + (f\,x + g\,y)\,d\,x = 0$.

Here

$$P = f\,x + g\,y,\ Q = a\,x + b\,y$$
$$U = f\,u + g,\ U' = a\,u + b$$
$$\therefore \frac{d\,y}{y} + \frac{(f\,u + g)\,d\,u}{f\,u^2 + (g + a)\,u + b} = 0$$

which being rational is Integrable by art. (88, 89)

Ex. 2. $x\,d\,y — y\,d\,x = d\,x\,\surd\,(x^2 + y^2)$

Here

$$Q = x,\ P = —y — \surd\,(x^2 + y^2)$$
$$U' = u,\ U = —1 — \surd\,(1 + u^2)$$
$$\therefore \frac{d\,y}{y} + \frac{1 + \surd\,(1 + u^2)}{u\,\surd\,(1 + u^2)}\,d\,u = 0$$

or

$$\frac{d\,y}{y} + \frac{d\,u}{u} + \frac{d\,u}{u\,\surd\,(1 + u^2)} = 0$$

which is Integrable by art. (82, 85.)

These Forms are called *Homogeneous.*

121. *To Integrate*

$$(a\,x + b\,y + c)\,d\,y + (m\,x + n\,y + p)\,d\,x = 0.$$

By assuming

$$\left.\begin{array}{r} a\,x + b\,y + c = u \\ \text{and} \qquad m\,x + n\,y + p = v \end{array}\right\}$$

we get

$$d\,y = \frac{m\,d\,u — a\,d\,v}{m\,b — n\,a},\ \text{and}\ d\,x = \frac{b\,d\,v — n\,d\,u}{m\,b — n\,a}$$

and therefore

$$(m\,u — n\,v)\,d\,u + (b\,v — a\,u)\,d\,u = 0$$

which being *Homogeneous* is Integrable by Art. 120.

We now come to that class of Integrals which is of the greatest use in Natural Philosophy—to

LINEAR EQUATIONS.

122. *Required to Integrate*

$$d y + y X d x = X' d x,$$

where X, X′ are functions of X.

Let

$$y = u v.$$

Then

$$u d v + v d u + X u v d x = X' d x$$

Hence assuming

$$d v + v X d x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

we have also

$$v d u = X' d x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Hence

$$\frac{d v}{v} + X d x = 0$$

$$\therefore l v + \int X d x = C$$

or

$$v = e^{C - \int X d x}$$
$$= e^{C} \times e^{-\int X d x}$$
$$= C \times e^{-\int X d x}.$$

Substituting for v in (b) we therefore get

$$d u = \frac{1}{C} . e^{\int X d x} X' d x$$

which may be Integrated in many cases by Art. 118.

Ex. $d y + y d x = a x^3 d x$.

Here

$$X = 1, X' = a x^3$$
$$\int X d x = x$$

and

$$\int X' d x \, e^{\int X d x} = a \int x^3 e^x d x$$
$$= a e^x (x^3 - 3 x^2 + 6 x - 6)$$

see Art. (102)

Hence

$$y = C e^{-x} + a (x^3 - 3 x^2 + 6 x - 6)$$

122. *Required to Integrate the* LINEAR *Equation of the second order*

$$\frac{d^2 y}{d x^2} + X \frac{d y}{d x} + X' y = 0$$

where X, X′ *are functions of* x.

Let $y = e^{\int u d x}$; then $\frac{d y}{d x} = u e^{\int u d x}$

$$\frac{d^2 y}{d x^2} = e^{\int u d x} \left(\frac{d u}{d x} + u^2\right)$$

and ∴ by substitution,

$$\frac{d u}{d x} + u^2 + X u + X' = 0$$

which is an equation of the first order and in certain cases may be Integrable by some one of the preceding methods. When for instance X and X′ are constants and a, b roots of the equation

$$u^2 + X u + X' = 0$$

then it will be found that

$$y = C e^{a x} + C' e^{b x}.$$

123. *Required the Integral of*

$$\frac{d^2 y}{d x^2} + X \frac{d y}{d x} + X' y = X''$$

where X″ *is a new function of* x.

Let $y = t z$; then Differencing, and substituting, we may assume the result

$$\frac{d^2 z}{d x^2} + X \frac{d z}{d x} + X' z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and

$$\therefore d \left(\frac{d t}{d x}\right) + \left(\frac{d t}{d x}\right) \left(X + \frac{z}{Z} \cdot \frac{d z}{d x}\right) d x = \frac{X''}{z} \quad . \quad . \quad . \quad (b)$$

Hence (by 122) deriving z from (a) and substituting in (b) we have a *Linear Equation* of the first order in terms of $\left(\frac{d t}{d x}\right)$; whence $\left(\frac{d t}{d x}\right)$ may be found; and we shall thus finally obtain

$$y = t z = z \int \left(\frac{d x}{z e^{2 \int X d x}} \int X'' e^{\int X d x} z d x\right).$$

Ex. $\frac{d^2 y}{d x^2} + \frac{d y}{d x} \cdot \frac{1}{x} - \frac{1}{x^2} \cdot y = \frac{a}{x^2 - 1}$.

Here

$$X = \frac{1}{x}, \; X' = -\frac{1}{x^2}, \; X'' = \frac{a}{x^2 - 1}.$$

Equat. (a) becomes

$$\frac{d^2 z}{d x^2} + \frac{d z}{d x} \cdot \frac{1}{x} = \frac{z}{x^2}$$

whence

$$d u + \left(u^2 + \frac{u}{x} - \frac{1}{x^2}\right) d x = 0$$

wherein $z = e^{\int u d x}$; which becomes *homogeneous* when for u we put v^{-1}. Next the variables are separated by putting (see 120)

$$x = v s$$

and we have

$$\frac{d v}{v} + \frac{s^2 + s - 1}{s (s^2 - 1)} d s$$

and

$$\therefore v = \frac{1}{s} \sqrt{\frac{s + 1}{s - 1}}.$$

Hence

$$u = \frac{x^2 + 1}{x (x^2 - 1)}, \int u d x = l . \frac{x^2 - 1}{x}$$

and

$$z = e^{\int u d x} = \frac{x^2 - 1}{x}.$$

Again

$$e^{\int X d x} = e^{l x} = x$$

and

$$\int X'' e^{\int X d x} z d x = \int a d x = a x + C$$

and

$$y = \frac{x^2 - 1}{x} \int \frac{(a x + C) x d x}{(x^2 - 1)^2};$$

which being Rational may be farther integrated, and it is found that finally

$$y = - \frac{a x + C}{2 x} + \frac{x^2 - 1}{4 x} a l . \left(C' . \frac{x - 1}{x + 1}\right).$$

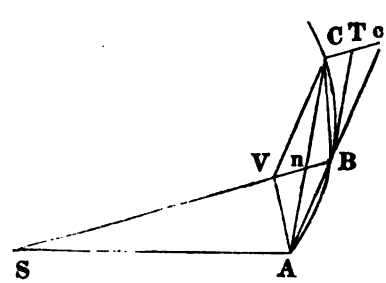
Here we shall terminate our long digression. We have exposed both the Direct and Inverse Calculus sufficiently to make it easy for the reader to comprehend the uses we may hereafter make of them, which was the main object we had in view. Without the Integral Calculus, in some shape or other, it is impossible to prosecute researches in the *higher* branches of philosophy with any chance of success; and we accordingly see Newton, partial as he seems to have been of Geometrical Synthesis, frequently have recourse to its assistance. His Commentators, especially

the Jesuits Le Seur and Jacquier, and Madame Chastellet (or rather Clairaut), have availed themselves on all occasions of its powers. The reader may anticipate, from the trouble we have given ourselves in establishing its rules and formulæ, that we also shall not be very scrupulous in that respect. Our design is, however, not perhaps exactly as he may suspect. As far as the Geometrical Methods will suffice for the comments we may have to offer, so far shall we use them. But if by the use of the Algorithmic Formulæ any additional truths can be elicited, or any illustrations given to the text, we shall adopt them without hesitation.

SECTION II. PROP. I.

124. This Proposition is a generalization of the Law discovered by Kepler from the observations of Tycho Brahe upon the motions of the planets and the satellites.

"*When the body has arrived at* B," says Newton, "*let a centripetal force act at once with a strong impulse, &c.*"] But were the force acting *incessantly* the body will arrive in the next instant at the same point C.



For supposing the centripetal force incessant, the path of the body will evidently be a curve such as A B C. Again, if the body move in the chord A B, and A B, B C be chords described in equal times, the deflection from A B, produced by an impulsive force acting only at B and communicating a velocity which would have been generated by the incessant force in the time through A B, is C c. But if the force had been incessant instead of impulsive, the body would have been moving in the tangent B T at B, and in this case the deflection at the end of the time through B C would have been half the space described with the whole velocity generated through B C (Wood's Mech.) But

$$C T = \tfrac{1}{2} C c$$

$\therefore$ the body would still be at C.

AN ANALYTICAL PROOF.

Let F denote the central force tending constantly to S (see Newton's figure), which take as the origin of the rectangular coordinates (x, y) which determine the place the body is in at the end of the time t. Also let ϱ be the distance of the body at that time from S, and θ the angular distance of ϱ from the axis of x. Then F being resolved parallel to the axis of x, y, its components are

$$F \cdot \frac{x}{\varrho} \text{ and } F \cdot \frac{y}{\varrho}$$

and (see Art. 46) we ∴ have

$$\frac{d^2 x}{d t^2} = - F \cdot \frac{x}{\varrho}, \ \frac{d^2 y}{d t^2} = - F \cdot \frac{y}{\varrho}.$$

Hence

$$\frac{y d^2 x}{d t^2} = - F \cdot \frac{x y}{\varrho} = \frac{x d^2 y}{d t^2}$$

$$\therefore \frac{y d^2 x - x d^2 y}{d t} = 0.$$

But

$$y d^2 x - x d^2 y = d y d x + y d^2 x - d x d y - x d^2 y$$
$$= d . (y d x - x d y)$$

∴ integrating

$$\frac{y d x - x d y}{d t} = \text{constant} = c.$$

Again,

$$x = \varrho \cos. \theta, \ y = \varrho \sin \theta, \ x^2 + y^2 = \varrho^2$$
$$\therefore d x = - \varrho d \theta \sin. \theta + d \varrho \cos. \theta$$
$$d y = \varrho d \theta \cos. \theta + d \varrho \sin. \theta;$$

whence by substitution we get

$$y d x - x d y = \varrho^2 d \theta$$
$$\therefore \frac{\varrho^2 d \theta}{d t} = c$$

But (see Art. 113)

$$\frac{\varrho^2 d \theta}{2} = d . (\text{Area of the curve}) = d . A$$

$$\therefore d t = \frac{\varrho^2 d \theta}{c} = \frac{2}{c} . d A.$$

Now since the time and area commence together in the integration there is no constant to be added.

$$\therefore t = \frac{2}{c} \times A \propto A.$$

Q. e. d.

125. COR. 1. PROP. II. By the comment upon LEMMA X, it appears that generally

$$v = \frac{d s}{d t}$$

and here, since the times of describing A B, B C, &c. are the same by hypothesis, d t is given. Consequently

$$v \propto d s$$

that is the velocities at the points A, B, C, &c. are as the elemental spaces described A B, B C, C D, &c. respectively. But since the area of a $\triangle$ generally = semi-base $\times$ perpendicular, we have, in symbols,

$$d. A = p \times d s$$

$$\therefore v \propto d s \propto \frac{d. A}{p};$$

and since the $\triangle$ A B S, B C S, C D S, &c. are all equal, d A is constant, and we finally get

$$v \propto \frac{1}{p} \text{ or } = \frac{c}{p}$$

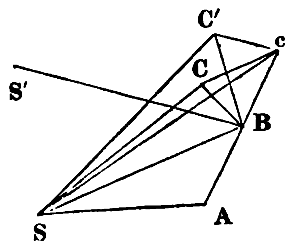
the constant being determinable, as will be shown presently, from the nature of the curve described and the *absolute* attracting force of S.

126. COR. 2. The parallelogram C A being constructed, C V is equal and parallel to A B. But A B = B c by construction and they are in the same line. Therefore C V is equal and parallel to B c. Hence B V is parallel to C c. But S B is also parallel to C c by construction, and B V, B S have one point in common, viz. B. They therefore coincide. That is B V, when produced passes through S.

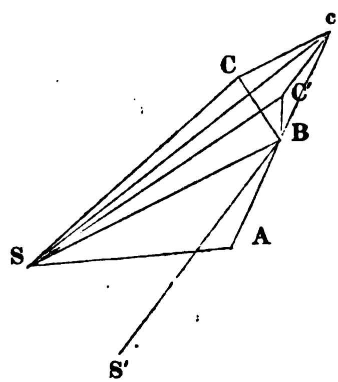
127. COR. 3. The body when at B is acted on by two forces; one in the direction B c, the momentum which is measured by the product of its mass and velocity, and the other the attracting *single* impulse in the direction B S. These acting for an instant produce by composition the momentum in the direction B C measurable by the actual velocity $\times$ mass. Now these component and compound momentums being each proportional to the product of the mass and the initial velocity of the body in the directions B c, B V, and B C respectively, will be also proportional to their initial velocities simply, and therefore by (125) to B V, B c, B C.

Hence B V measures the force which attracts the body towards S when the body is at B—and so on for every other position of the body.

128. Cor. 1. Prop. II. In the annexed figure B c = A B, C c is parallel to S B, and C′ c is parallel to S′ B. Now △ S C B = S c B = S A B, and if the body by an impulse of S have deflected from its rectilinear course so as to be in C, by the proposition the direction in which the centripetal force acts is that of C c or S B. But if, the body having arrived at C′, the △ S B C′ be > S A B (the times of description are equal by hypothesis) and ∴ > S B C, the vertex C′ falls without the △ S B C, and the direction of the force along c C′ or B S′, has clearly declined from the course B S in consequentia.



The other case is readily understood from this other diagram.



129. *To prove that a body cannot describe areas proportional to the times round two centers.*

If possible let

$$\triangle S' A B = \triangle S' B C$$

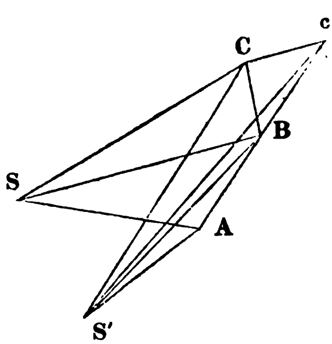
and

$$S A B = S B C.$$

Then

$$\triangle S' B C (= S' A B) = S' B c$$

and C c is parallel to S′ B. But it is also parallel to S B by construction. Therefore S B and S′ B coincide, which is contrary to hypothesis.



130. Prop. III. The demonstration of this proposition, although strictly rigorous, is rather puzzling to those who read it for the first time. At least so I have found it in instruction. It will perhaps be clearer when stated symbolically thus:

Let the central body be called T and the revolving one L. Also let the whole force on L be F, its centripetal force be f, and the force ac-

celerating T be f'. Then supposing a force equal to f' to be applied to L and T in a direction opposite to that of f', by COR. 6. of the Laws, the force f will cause the body L to revolve as before, and we have remaining

$$f = F - f'$$

or

$$F = f + f'.$$

Q. e. d.

ILLUSTRATION.

Suppose on the deck of a vessel in motion, you whirl a body round in a vertical or other plane by means of a string, it is evident the centrifugal force or tension of the string or the power of the hand which counteracts that centrifugal force—i. e. the centripetal force will not be altered by the force which impels the vessel. Now the motion of the vessel gives an equal one to the hand and body and in the same direction; therefore the force on the body = force on the hand + centripetal power of the hand.

131. PROP. IV. Since the motion of the body in a circle is uniform by supposition, the arcs described are proportional to the times. Hence

$$t \propto \text{arc described} \propto \frac{\text{arc} \times \text{radius}}{2}$$

$$\propto \text{area of the sector.}$$

Consequently by PROP. II. the force tends to the center of the circle.

Again the motion being equable and the body always at the same distance from the center of attraction, the centripetal force (F) will clearly be every where the same in the same circle (see COR. 3. PROP. I.) But the absolute value of the force is thus obtained.

Let the arc A B (fig. in the Glasgow edit.) be described in the time T. Then by the centripetal force F, (which supposing A B indefinitely small, may be considered constant,) the sagitta D B (S) will be described in that time, and (Wood's Mechanics) comparing this force with gravity as the unit of force put = 1, we have

$$S = \frac{g}{2} F T^2$$

$$g \text{ being} \doteq 32\tfrac{1}{6} \text{ feet.}$$

But by similar triangles A B D, A B G

$$S = B D = \frac{(\text{chord } A B)^2}{A G} = \frac{(\text{arc } A B)^2}{2 R} \text{ ult}^y.$$

(Lemma VII.)

$$\therefore F = \frac{2S}{gT^2} = \frac{(\text{arc } AB)^2}{gRT^2}.$$

If T be given

$$F \propto \frac{(\text{arc } AB)^2}{R}.$$

If T = arc second

$$F = \frac{(\text{arc } AB)^2}{gR}.$$

132. Cor. 1. Since the motion is uniform, the velocity is

$$v = \frac{\text{arc}}{T}$$

$$\therefore F = \frac{v^2}{gR} \propto \frac{v^2}{R}.$$

133. Cor. 2. The Periodic Time is

$$P = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi R}{v}$$

$$\therefore F = \frac{4\pi^2 R^2}{gRP^2} = \frac{4\pi^2 R}{gP^2} \propto \frac{R}{P^2}.$$

134. Cor. 3, 4, 5, 6, 7. Generally let

$$P = k \times R^n,$$

k being a constant.
Then

$$v = \frac{2\pi R}{P} = \frac{2\pi}{kR^{n-1}} \propto \frac{1}{R^{n-1}}$$

and

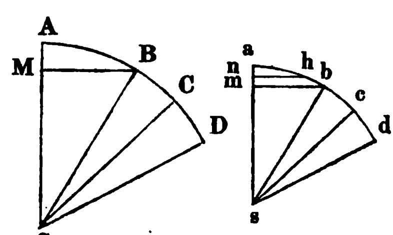
$$F = \frac{4\pi^2 R}{gk^2 R^{2n}} = \frac{4\pi^2}{gk^2 R^{2n-1}} \propto \frac{1}{R^{2n-1}}.$$

Conversely. If $F \propto \frac{1}{R^{2n-1}}$; P will $\propto R^n$.

For (133)

$$P \propto \sqrt{\frac{R}{F}} \propto \sqrt{R^{2n}} \propto R^n.$$

135. Cor. 8. A B, a b are similar arcs, and A B, a h contemporaneously described and indefinitely small.



Now ultimately

$$an : am :: ah^2 : ab^2$$

and

$$am : AM :: ab : AB$$

(Lemma V)

$$\therefore an : AM :: ah^2 : ab . AB$$

or

$$f : F :: \frac{a h^2}{a b} : \frac{A B^2}{A B} :: \frac{a h^2}{a s} : \frac{A B^2}{A S} \text{ (LEMMA V)}$$

$$:: \frac{v^2}{a s} : \frac{V^2}{A S}$$

or

$$F \propto \frac{V^2}{A S}.$$

And if the whole similar curves A D, a d be divided into an equal number of indefinitely small equal areas A B S, B C S, &c.; a b s, b c s, &c. these will be similar, and, by composition of ratios, (P and p being the whole times)

$$P : p :: \text{time through A B} : \text{time through a b}$$

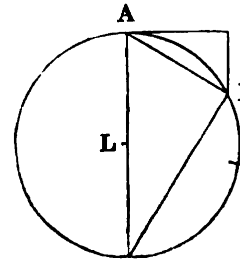
$$:: \frac{A B}{V} : \frac{a b}{v} :: \frac{A S}{V} : \frac{a s}{v}$$

$$\therefore P \propto \frac{A S}{V}.$$

Hence

$$F \propto \frac{V^2}{A S} \propto \frac{A S}{P^2}.$$

136. COR. 9. Let A C be uniformly described, and with the force considered constant, suppose the body would fall to L in the same time in which it would revolve to C. Then A B being indefinitely small, the force down R B may be considered constant, and we have (131)



$$A C^2 : A B^2 :: T^2_{AB} : T^2_{AC}$$

$$:: T^2_{AL} : T^2_{RB}$$

$$:: AL : RB \text{ (131)}$$

$$:: AL : \frac{AB^2}{AD}$$

Hence

$$A B^2 = A L \times A D.$$

PROP. VI. Sagitta $\propto$ F when time is given. Also sag. $\propto$ (arc)^2 by LEMMA XI, $\propto t^2$ when F is given

$\therefore$ when neither force nor time is given

$$\text{sag.} \propto F \times t^2;$$

$$\therefore F \propto \frac{\text{sag.}}{t^2}$$

OTHERWISE.

By Lemma X, Cor. 4,

$$F \propto \frac{\text{space ipso motus initio}}{t^2}$$

$$\propto \frac{\text{sag.}}{t^2}.$$

To generalize this expression, let $\frac{g}{2}$ be the space described in $1''$ at the surface of the Earth by Gravity. Also let the unit of force be Gravity. Then

$$F : 1 :: \frac{\text{sag.}}{t^2} : \frac{g}{2 \times 1''^2}$$

$$\therefore F = \frac{2\,\text{sag.}}{g\,t^2} = \frac{2}{g} \times \frac{s}{t^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(a)}$$

by hypothesis.

137. Cor. 1. $F \propto \frac{QR}{t^2} \propto \frac{QR}{(\text{area } SPQ)^2}$

$$\propto \frac{QR}{SP^2 \times QT^2}.$$

To generalize this, let a be the area described in $1''$. Then the area described in $t'' = a \times t = \frac{SP \times QT}{2}$.

$$\therefore t = \frac{SP \times QT}{2a};$$

and substituting in (a) we get

$$F = \frac{8a^2}{g} \times \frac{QR}{SP^2 \times QT^2} \quad . \; . \; . \; . \; . \; . \quad \text{(b)}$$

Again, if the Trajectories turn into themselves, there must be

$$a : 1'' :: A \text{ (whole Area)} : T \text{ (Period. Time)}$$

$$\therefore a = \frac{A}{T}.$$

Hence by (b) we have

$$F = \frac{8A^2}{gT^2} \times \frac{QR}{SP^2 \times QT^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(c)}$$

which, in practice, is the most convenient expression.

138. Cor. 2. $F = \frac{8A^2}{gT^2} \times \frac{QR}{SY^2 \times QP^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(d)}$

139. Cor. 3. $F = \frac{8A^2}{gT^2} \times \frac{1}{SY^2 \times PV} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(e)}$

Hence is got a differential expression for the force. Since

$$P V = \frac{2 p d \varrho}{d p}$$

$$\therefore F = \frac{8 A^2}{g T^2} \times \frac{1}{\frac{2 p^2 p d \varrho}{d p}}$$

$$= \frac{4 A^2}{g T^2} \times \frac{d p}{p^3 d \varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Another is the following in terms of the reciprocal of the Radius Vector ϱ and the traced-angle θ.

Because

$$p = \frac{\varrho^2 d \theta}{\sqrt{(d \varrho^2 + \varrho^2 d \theta^2)}},$$

$$\therefore \frac{1}{p^2} = \frac{d \varrho^2 + \varrho^2 d \theta^2}{\varrho^4 d \theta^2}$$

$$= \frac{d \varrho^2}{\varrho^4 d \theta^2} + \frac{1}{\varrho^2}.$$

Let $$\frac{1}{\varrho} = u.$$

Then

$$d \varrho = - \frac{d u}{u^2}$$

also

$$\frac{1}{p^2} = \frac{d u^2}{d \theta^2} + u^2$$

$$\therefore - \frac{2 d p}{p^3} = \frac{2 d u d^2 u}{d \theta^2} + 2 u d u$$

$$\therefore \frac{d p}{p^3 d \varrho} = \frac{d^2 u}{d \theta^2} u^2 + u^3$$

and substituting in f we have

$$F = \frac{4 A^2}{g T^2} \times \left(\frac{d^2 u}{d \theta^2} u^2 + u^3\right) \quad . \quad . \quad . \quad . \quad . \quad (g)$$

140. COR. 4. $$F \propto \frac{1}{S Y^2} \times \frac{1}{P V} \propto V^2 \times \frac{1}{P V}$$

$$\propto \frac{V^2}{P V}.$$

This is generalized thus. Since

$$V = \frac{\text{space}}{\text{Time}} = \frac{P Q}{t}$$

and

$$a \times t \left(= \frac{A}{T} \times t\right) = \text{area described}$$

$$= \frac{PQ \times SY}{2}$$

$$\therefore V = \frac{PQ}{t} = \frac{2A}{T} \times \frac{1}{SY}.$$

Hence

$$\frac{1}{SY^2} = \frac{T^2}{4A^2} \times V^2$$

and by Cor. 3.

$$F = \frac{2}{g} \times \frac{V^2}{PV} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

From this formula we get

$$V^2 = \frac{g}{2} \times F \times PV$$

$$= 2gF \times \frac{PV}{4}.$$

But by Mechanics, if s denote the space moved through by a body urged by a constant force F

$$V^2 = 2gF \times s$$

$$\therefore s = \frac{PV}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

that is, *the space through which a body must fall when acted on by the force continued constant to acquire the velocity it has at any point of the Trajectory, is* $\frac{1}{4}$ *of the chord of curvature at that point.*

Also

$$V^2 = 2gF \times \frac{p\,d\rho}{2\,dp} = gF \times \frac{p\,d\rho}{dp} \quad . \quad . \quad . \quad . \quad (k)$$

The next four propositions are merely examples to the preceding formulæ.

141. Prop. VII.

$$RP^2 (= QR \times RL) : QT^2 :: AV^2 : PV^2$$

$$\therefore \frac{QR \times RL \times PV^2}{AV^2} = QT^2$$

and multiplying both sides by $\frac{SP^2}{QR}$ and putting P V for R L, we have

$$\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$$

$$\therefore F \propto \frac{AV^2}{SP^2 \times PV^3} \propto \frac{1}{SP^2 \times PV^3}.$$

Also by (137 c.)

$$F = \frac{8A^2}{gT^2} \times \frac{AV^2}{SP^2 \times PV^3} = \frac{32\pi r^4}{gT^2} \times \frac{1}{SP^2 \times PV^3}.$$

OTHERWISE.

From similar triangles we get

$$AV : PV :: SP : SY$$

$$\therefore SY = \frac{SP \times PV}{AV}.$$

$$\therefore SY^2 \times PV = \frac{SP^2 \times PV^2}{AV^2} \times PV$$

$$= \frac{SP^2 \times PV^3}{AV^2}$$

$$\therefore F \propto \frac{1}{SY^2 \times PV} \propto \frac{1}{SP^2 \times PV^3}$$

as before.

OTHERWISE.

$$p = \frac{r^2 - a^2 + \rho^2}{2r}$$

is the equation to the circle; whence

$$\frac{dp}{d\rho} = \frac{\rho}{r}$$

$$\therefore F = \frac{4A^2}{gT^2} \times \frac{dp}{p^3 d\rho} = \frac{4A^2}{gT^2} \times \frac{\rho}{rp^3}$$

$$= \frac{4\pi r}{gT^2} \times \rho \times \frac{8r^3}{(r^2 - a^2 + \rho^2)^3}$$

$$= \frac{32\pi r^4}{gT^2} \times \frac{\rho}{(r^2 - a^2 + \rho^2)^3}.$$

OTHERWISE.

The polar equation to the circle is

$$\rho = \frac{2a \cos. \theta}{1 + \cos.^2 \theta}$$

$$\therefore u \left(= \frac{1}{\rho}\right) = \frac{1}{2a \cos. \theta} + \frac{\cos. \theta}{2a}$$

$$\therefore \frac{du}{d\theta} = \frac{1}{2a}\left(\frac{\sin. \theta}{\cos.^2 \theta} - \sin. \theta\right)$$

$$= \frac{1}{2a} \times \frac{\sin.^3 \theta}{\cos.^2 \theta}$$

$$\therefore \frac{d^2 u}{d\theta^2} = \frac{1}{2a}\left(\frac{3 \sin.^2 \theta}{\cos. \theta} + \frac{2 \sin^4 \theta}{\cos.^3 \theta}\right)$$

$$= \frac{1}{2a} \times \frac{\sin.^2 \theta}{\cos.^3 \theta} \times (3 - \sin.^2 \theta).$$

Hence

$$\frac{d^2 u}{d\theta^2} + u = \frac{\sin.^2\theta}{2a\cos.^3\theta}\cdot(3-\sin^2\theta) + \frac{1}{2a\cos.\theta} + \frac{\cos.\theta}{2a}$$

$$= \frac{1}{2a\cos.^3\theta}\times(3\sin^2\theta - \sin.^4\theta + \cos.^2\theta + \cos.^4\theta)$$

$$= \frac{1}{2a\cos.^3\theta}\times(2\sin.^2\theta - \sin.^4\theta + 1 + 1 - 2\sin.^2\theta + \sin.^4\theta)$$

$$= \frac{1}{a\cos.^3\theta}.$$

which by (139) gives

$$F = \frac{4A^2}{gT^2}\times\frac{u^2}{a\cos.^3\theta}$$

$$= \frac{4A^2}{gT^2}\times\frac{(1+\cos.^2\theta)^2}{4a^3\cos.^5\theta}$$

$$= \frac{A^2}{ga^3T^2}\times\frac{(1+\cos.^2\theta)^2}{\cos.^5\theta}.$$

142. Cor. 1. $F \propto \frac{1}{SP^2 \times PV^3}$.

But in this case

$$SP = PV.$$

$$\therefore F \propto \frac{1}{SP^5}, \text{ or } = \frac{32\pi r^4}{gT^2}\times\frac{1}{SP^5}.$$

Cor. 2. $\underset{S}{F} : \underset{R}{F} :: RP^2 \times PT^3 : SP^2 \times PV^3$

$$:: SP \times RP^2 : \frac{SP^3 \times PV^3}{PT^3}$$

$$:: SP \times RP^2 : SG^3,$$

by similar triangles.

This is true when the periodic times are the same. When they are different we have

$$\underset{S}{F} : \underset{R}{F} :: SP \times RP^2 \frac{\underset{S}{T}}{\underset{R}{T}} \times SG^3,$$

where the notation explains itself.

143. Prop. VIII.

$$CP^2 : PM^2 :: PR^2 : QT^2$$

and

$$PR^2 = QR \times (RN + QN) = QR \times 2PM$$

$$\therefore CP^2 : PM^2 :: QR \times 2PM : QT^2$$

$$\therefore \frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$$

and

$$\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}$$

$$\therefore F \propto \frac{CP^2}{2PM^3 \times SP^2} \propto \frac{1}{PM^3}.$$

Also by 137,

$$F = \frac{4a^2}{g} \times \frac{CP^2}{SP^2 \times PM^3}.$$

But

$$a = \frac{SP \times \text{velocity}}{2} = \frac{SP \times V}{2}$$

$$\therefore F = \frac{V^2}{g} \times \frac{CP^2}{PM^3}.$$

OTHERWISE.

By PROP. VII,

$$F \propto \frac{1}{SP^2 \times PV^3}.$$

But S P is infinite and P V = 2 P M.

$$\therefore F \propto \frac{1}{PM^3}.$$

OTHERWISE.

The equation to the circle from any point without it is

$$p = \frac{c^2 - r^2 - \varrho^2}{2r}$$

where c is the distance of the point from the center, and r the radius.

$$\therefore \frac{dp}{d\varrho} = -\frac{\varrho}{r}$$

Moreover in this case

$$\varrho = c + PM = c + y$$

$$\therefore p = \frac{c^2 - r^2 - c^2 - 2cy - y^2}{2r}$$

$$= -\frac{cy}{r}$$

$$\therefore \frac{dp}{p^3 d\varrho} = \frac{c+y}{r} \times \frac{r^3}{c^3 y^3}$$

$$= \frac{r^2}{c^2 y^3}.$$

Hence (139)

$$F = \frac{4a^2 r^2}{c^2 g} \times \frac{1}{y^3} = \frac{V^2 r^2}{g} \times \frac{1}{y^3}.$$

SCHOLIUM.

144. Generally we have

$$PR^2 : QT^2 :: PC^2 : PM^2$$

$$\therefore \frac{PR^2}{QR} : \frac{QT^2}{QR} :: PC^2 : PM^2$$

But

$$\frac{PR^2}{QR} = PV,$$

and

$$PC : PM :: 2R \text{ (R = rad. of curvature)} : PV$$

$$\therefore \frac{QT^2}{QR} = PV \times \frac{PM^2}{PC^2} = \frac{2R \times PM}{PC}$$

$$= \frac{2R \times PM^3}{PC^3}.$$

But

$$R = \frac{AC^2}{BC^4} \times PC^3$$

$$\therefore \frac{QT^2}{QR} = \frac{2AC^2}{BC^4} \times PM^3$$

and

$$F \propto \frac{1}{PM^3}.$$

From the expression (g. 139) we get

$$F = \frac{4a^2}{g} \times \frac{d^2 u}{d\theta^2} \times u^2.$$

But

$$a \times t = \frac{\varrho^2 d\theta}{2} = a \times \frac{dx}{V}$$

$$\therefore \frac{4a^2}{d\theta^2} = \frac{V^2 \varrho^4}{dx^2}.$$

Also

$$u = \frac{1}{\varrho}.$$

$$\therefore du = -\frac{d\varrho}{\varrho^2}$$

and

$$d^2 u = -\frac{d^2 \varrho}{\varrho^2} + \frac{2 d \varrho^2}{\varrho^3}$$
$$= -\frac{d^2 \varrho}{\varrho^2} \text{ (see 69)}$$

Hence

$$F = \frac{V^2 \varrho^4}{g d x^2} \times -\frac{d^2 \varrho}{\varrho^2} \times \frac{1}{\varrho^2}$$
$$= -\frac{V^2}{g} \times \frac{d^2 \varrho}{d x^2}$$
$$= -\frac{V^2}{g} \times \frac{d^2 y}{d x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

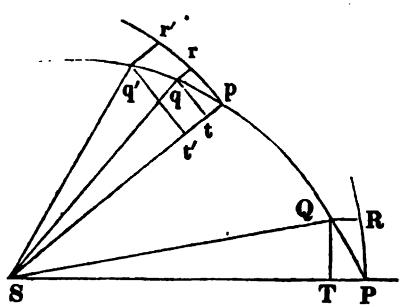
This is moreover to be obtained at once from (see 48)

$$F = \frac{1}{g} \times \frac{d^2 y}{d t^2}.$$

For

$$d t = \frac{d s}{V}$$
$$\therefore F = \frac{V^2}{g} \times -\frac{d^2 y}{d x^2}.$$

145. PROP. IX. Another demonstration is the following:



Let $\angle$ P S Q = $\angle$ p S q. Then from the nature of the spiral the angles at P, Q, p, q being all equal, the triangles S P Q, S p q are similar. Also we have the triangles R P Q, r p q similar, as likewise Q P T, q p t.

Hence

$$\frac{QT^2}{QR} : \frac{q t^2}{q r} :: SP : Sp$$

and by LEMMA IX.

$$q' r : q r :: p r'^2 : p r^2 :: q' t'^2 : q t^2$$

$$\therefore \frac{q' t'^2}{q' r'} = \frac{q t^2}{q r}.$$

Hence

$$\frac{q' t'^2}{q' r'} : \frac{Q T^2}{Q R} :: S p : S P$$

$$\therefore \frac{Q T^2}{Q R} \propto S P$$

and

$$\frac{Q T^2 \times S P^2}{Q R} \propto S P^3$$

$$\therefore F \propto \frac{1}{S P^3}.$$

OTHERWISE.

The equation to the logarithmic spiral is

$$p = \frac{b}{a} \times \varrho$$

$$\therefore \frac{d p}{d \varrho} = \frac{b}{a}$$

and by (f. 139) we have

$$F = \frac{4 \alpha^2}{g} \times \frac{d p}{p^3 d \varrho} = \frac{4 \alpha^2}{g} \times \frac{b}{a} \times \frac{a^3}{b^3 \varrho^3}$$

$$= \frac{4 \alpha^2 . a^2}{g . b^2} \times \frac{1}{\varrho^3}.$$

Using the polar equation, viz.

$$\theta = \frac{b}{\sqrt{(a^2 - b^2)}} \times \log. \frac{\varrho}{a}$$

the force may also be found by the formula (g).

146. Prop. X.

$$\left. \begin{array}{r} P v \times v G : Q v^2 :: P C^2 : C D^2 \\ Q v^2 : Q T^2 :: P C^2 : P F^2 \end{array} \right\}$$

$$\therefore P v \times v G : Q T^2 :: P C^4 : C D^2 \times P F^2$$

$$\therefore v G : \frac{Q T^2}{P v} :: P C^2 : \frac{C D^2 \times P F^2}{P C^2}$$

But

$$P V = Q R, \text{ and } C D \times P F = \text{(by Conics) } B C \times C A$$

also

$$\text{ult. } v G = 2 P C.$$

$$\therefore 2 P C : \frac{Q T^2}{Q R} :: P C^2 : \frac{B C^2 \times C A^2}{P C^2}$$

$$\therefore F \propto \frac{QR}{QT^2 \times CP^2} \propto \frac{PC}{2BC^2 \times CA^2} \propto PC.$$

Also by expression (c. 137) we get

$$F = \frac{8A^2}{gT^2} \times \frac{PC}{2BC^2 \times CA^2}$$

But

$$A = \pi \times BC \times CA$$

$$\therefore F = \frac{4\pi^2}{gT^2} \times PC.$$

The additional figure represents an Hyberbola. The same reasoning shows that the force, being in the center and *repulsive*, also in this curve, $\propto CP$.

ALITER.

Take

$$Tu = Tv$$

and

$$uV : vG :: DC^2 : PC^2$$

Then since

$$Qv^2 : Pv \times vG :: DC^2 : PC^2$$

$$\therefore uV : vG :: Qv^2 : Pv \times vG$$

$$\therefore Qv^2 = Pv \times uV$$

$$\therefore Qv^2 + uP \times Pv = Pv \times (uV + uP)$$

$$= Pv \times VP.$$

But

$$Qv^2 = QT^2 + Tv^2 = QT^2 + Tu^2$$

$$= PQ^2 - PT^2 + Tu^2$$

$$= PQ^2 - (PT^2 - Tu^2)$$

$$= PQ^2 - Pu \times Pv$$

$$(\text{chord } PQ)^2 = Pv \times VP.$$

Now suppose a circle touching P R in P and passing through Q to cut P G in some point V'. Then if Q V' be joined we have

$$\angle PQv = \angle QPR = \angle QV'P$$

and in the $\triangle QPv$, $QV'P$ the $\angle QPV'$ is common. They are therefore similar, and we have

$$Pv : PQ :: PQ : PV'$$

$$\therefore PQ^2 = Pv \times V'P = Pv \times VP$$

$$\therefore V'P = VP$$

or the circle in question passes through V;

$\therefore$ P V is the chord of curvature passing through C.

Again, since

$$u V = v G \times \frac{D C^2}{P C^2} = C \times v G$$

or

$$P V - P u = C (P G - P v)$$

and

$$P V, P G$$

being homogeneous

$$P V = \frac{2 D C^2}{P C^2} P C = \frac{2 C D^2}{P C}.$$

∴ (Cor. 3, Prop. VI.)

$$F \propto \frac{P C}{2 P F^2 \times C D^2}.$$

But since by Conics the parallelogram described about an Ellipse is equal to the rectangle under its principal axes, it is constant. ∴ $P F \times C D$ is.

and

$$F \propto P C.$$

OTHERWISE.

By (f. 139) we have

$$F = \frac{4 A^2}{g T^2} \times \frac{d p}{p^3 d \rho}.$$

But in the ellipse referred to its center

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - \rho^2}$$

$$\therefore \frac{1}{p^2} = \frac{a^2 + b^2 - \rho^2}{a^2 b^2}$$

and differentiating, and dividing by -2, there results

$$\frac{d p}{p^3 d \rho} = \frac{\rho}{a^2 b^2}$$

which gives

$$F = \frac{4 A^2}{g T^2} \times \frac{\rho}{a^2 b^2} = \frac{4 \pi^2}{g T^2} \times \rho.$$

In like manner may the force be found from the polar equation to the ellipse, viz.

$$\rho^2 = \frac{b^2}{1 - e^2 \cos.^2 \theta},$$

by means of substituting in equat. (g. 139.)

147. COR. 1. For a geometrical proof of this converse, see the Jesuits' notes, or Thorpe's Commentary. An analytical one is the following.

Let the body at the distance R from the center be projected with the velocity V in a direction whose distance from the center of attraction is P. Also let

$$F = \mu \varrho$$

μ being the force at the distance 1. Then (by f)

$$F = \frac{4A^2}{gT^2} \times \frac{dp}{p^3 d\varrho} = \mu \varrho$$

which gives by integration, and reduction

$$\frac{1}{p^2} = \frac{\mu g T^2}{4A^2} \times R^2 + \frac{1}{P^2} - \frac{\mu g T^2}{4A^2} \times \varrho^2$$

R and P being corresponding values of ϱ and p.

But in the ellipse referred to its center we have

$$\frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} - \frac{\varrho^2}{a^2 b^2}$$

which shows that the orbit is also an ellipse with the force tending to its center, and equating homogeneous quantities, we get

$$\left.\begin{array}{l} \dfrac{a^2+b^2}{a^2 b^2} = \dfrac{\mu g T^2}{4A^2} \times R^2 + \dfrac{1}{P^2} \\ \text{and} \\ \dfrac{1}{a^2 b^2} = \dfrac{\mu g T^2}{4A^2} \end{array}\right\}$$

But

$$A = \pi a b$$

$$\therefore T = \frac{2\pi}{\sqrt{\mu g}} \quad \ldots\ldots\ldots\ldots \quad (1)$$

which gives the value of the periodic time, and also shows it to be constant. (See Cor. 2 to this Proposition.)

Having discovered that the orbit is an ellipse with the force tending to tne center, from the data, we can find the actual orbit by determining its semiaxes a and b.

By 140, we have

$$V = \frac{2A}{T} \times \frac{1}{P}$$

$$\therefore \frac{a^2+b^2}{a^2 b^2} = \mu g \times \frac{R^2}{V^2 P^2} + \frac{1}{P^2}$$

and

$$\frac{1}{a^2 b^2} = \mu g \times \frac{1}{V^2 P^2}$$

$$\therefore a^2 + b^2 = R^2 + \frac{V^2}{\mu g}.$$

and

$$2ab = \frac{2VP}{\sqrt{\mu g}}$$

$$\therefore a + b = \sqrt{\left(R^2 + \frac{V^2}{\mu g} + \frac{2VP}{\sqrt{\mu g}}\right)}$$

and

$$a - b = \sqrt{\left(R^2 + \frac{V^2}{\mu g} - \frac{2VP}{\sqrt{\mu g}}\right)}$$

which, by addition and subtraction, give a and b.

OTHERWISE.

By formula (g. 139,) we have

$$F = \frac{4A^2}{gT^2} u^2 \left(\frac{d^2 u}{d\theta^2} + u\right) = \mu \varrho = \frac{\mu}{u}$$

$$\therefore \frac{d^2 u}{d\theta^2} + u - \frac{g\mu T^2}{4A^2} \times \frac{1}{u^3} = 0$$

and multiplying by 2 d u, integrating and putting $\frac{g\mu T^2}{4A^2} = M$, we have

$$\frac{du^2}{d\theta^2} + u^2 + \frac{M}{u^2} + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To determine C, we have

$$\frac{du^2}{d\theta^2} = \frac{1}{\varrho^4} \cdot \frac{d\varrho^2}{d\theta^2}$$

and in all curves it is easily found that

$$\frac{d\varrho}{d\theta} = \frac{\varrho}{p} \sqrt{(\varrho^2 - p^2)}$$

$$\therefore \frac{du^2}{d\theta^2} = \frac{\varrho^2 - p^2}{\varrho^2 p^2} = \frac{1}{p^2} - \frac{1}{\varrho^2}.$$

Hence, when $\varrho = R$, and $p = P$,

$$\frac{1}{P^2} + MR^2 + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives the constant C.

Again from (2) we get

$$d\theta = \frac{u\,du}{\sqrt{(-M - Cu^2 - u^4)}}$$

which being integrated (see Hersch's Tables, p. 160.—Englished edit. published by Baynes & Son, Paternoster Row) and the constants properly determined will finally give ϱ in terms of θ; whence from the equation to the ellipse will be recognised the orbit and its dimensions.

148. COR. 2. This Cor. has already been demonstrated—see (1).

Newton's Proof may thus be rendered a little easier.

By Cor. 3 and 8 of Prop. IV, in *similar ellipses*

T is constant.

Again for Ellipses having the *same axis-major*, we have

$$T \left(\propto \frac{A}{\alpha}\right) \propto \frac{\pi a b}{\alpha} \propto \frac{b}{\alpha}.$$

But since the forces are the same at the principal vertexes, the sagittæ are equal, and ultimately the arcs, which measure the velocities, are equal to the ordinates, and these are as the axes-minores. Hence, α (which $= \frac{v \times SY}{2}$) $\propto$ b.

$$\therefore T \propto \frac{b}{b} \propto 1 \text{ or is constant.}$$

Again, generally if A and B be any two ellipses whatever, and C a third one similar to A, and having the same axis-major as B; then, by what has just been shown,

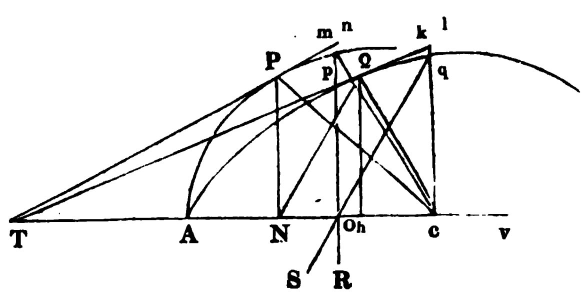
$$T \text{ in } B = T \text{ in } C$$

and

$$T \text{ in } C = T \text{ in } A$$

$$\therefore T \text{ in } B = T \text{ in } A.$$

149. SCHOL. See the Jesuits' Notes. Also take this proof of, "If one curve be related to another on the same axis by having its ordinates in a given ratio, and inclined at a given angle, the forces by which bodies are made to describe these curves in the *same time* about the same center in the axis are, in corresponding points, as the distances from the center."



The construction being intelligible from the figure, we have

$$PN : QN :: pO : qO$$

$$\therefore PN : pO :: QN \; qO$$

$$:: NT : OT \text{ ultimately.}$$

$\therefore$ Tangents meet in T,

the triangles C P T, C Q T are in the ratio of P N : Q h or of parallelograms P N O p, Q N O q ultimately, i. e. in the given ratio, and

$$\begin{aligned} \mathrm{CpP} : \mathrm{CPT} &:: \mathrm{pP} : \mathrm{PT} \text{ ultimately.} \\ &:: \mathrm{NO} : \mathrm{NT} \\ &:: \mathrm{qQ} : \mathrm{QT} \\ &:: \mathrm{CQq} : \mathrm{CQT} \end{aligned}$$

$\therefore$ C p P : C q Q in a given ratio.

$\therefore$ bodies describing equal areas in equal times, are in corresponding points at the same times.

$\therefore$ P p, Q q are described in the same time, and m p and k q are as the forces.

Draw C R, C S parallel to P T, Q T; then

$$\mathrm{pO} : \mathrm{qO} :: \mathrm{PN} : \mathrm{QN} :: \mathrm{nO} : \mathrm{lO}$$

$$\therefore \mathrm{nO} : \mathrm{pO} :: \mathrm{lO} : \mathrm{qO}$$

and

$$\left.\begin{aligned} \mathrm{np} : \mathrm{nO} &:: \mathrm{lq} : \mathrm{lQ} \\ \text{but} \quad \mathrm{nO} : \mathrm{nR} &:: \mathrm{lQ} : \mathrm{lS} \end{aligned}\right\}$$

(since $\mathrm{nO} : \mathrm{OR} :: \mathrm{TO} : \mathrm{OC} :: \mathrm{lO} : \mathrm{OS}$)

$$\therefore \mathrm{np} : \mathrm{nR} :: \mathrm{lq} : \mathrm{lS}$$

$$\therefore \mathrm{np} : \mathrm{pR} :: \mathrm{lq} : \mathrm{qS}$$

and

$$\left.\begin{aligned} \mathrm{np} : \mathrm{pR} &:: \mathrm{mp} : \mathrm{pC} \\ \mathrm{lq} : \mathrm{qS} &:: \mathrm{kq} : \mathrm{qC} \end{aligned}\right\}$$

$$\therefore \mathrm{mp} : \mathrm{pC} :: \mathrm{kq} : \mathrm{qC}$$

or

$$\mathrm{F\,at\,p} : \mathrm{F\,at\,q} :: \mathrm{pC} : \mathrm{qC}.$$

Q. e. d.

SECTION III.

150. Prop. XI. This proposition we shall simplify by arranging the proportions one under another as follows:

$$\begin{aligned} \mathrm{L} \times \mathrm{QR} (= \mathrm{Px}) : \mathrm{L} \times \mathrm{Pv} &:: \mathrm{PE} : \mathrm{PC} \\ &:: \mathrm{AC} : \mathrm{PC} \end{aligned}$$

But

$$\begin{aligned} \mathrm{L} \times \mathrm{Pv} &: \mathrm{Gv} \times \mathrm{Pv} :: \mathrm{L} : \mathrm{Gv} \\ \mathrm{Gv} \times \mathrm{Pv} &: \mathrm{Qv}^2 :: \mathrm{PC}^2 : \mathrm{CD}^2 \\ \mathrm{Qv}^2 &: \mathrm{Qx}^2 :: 1 : 1 \\ \mathrm{Qx}^2 &: \mathrm{QT}^2 :: \mathrm{PE}^2 : \mathrm{PF}^2 \\ & \qquad :: \mathrm{CA}^2 : \mathrm{PF}^2 \\ & \qquad :: \mathrm{CD}^2 : \mathrm{CB}^2 \end{aligned}$$

$\therefore L \times QR : QT^2 :: AC \times L \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2$
and

$$\frac{QR}{QT^2} = \frac{AC \times PC}{Gv \times CB^2} = \frac{AC \times PC}{2PC \times CB^2} = \frac{AC}{2CB^2}$$

$$\therefore F \propto \frac{QR}{QT^2 \times SP^2}\left(= \frac{AC}{2CB^2 \times SP^2}\right) \propto \frac{1}{SP^2}.$$

Q. e. d.

Hence, by expression (c) Art. 137, we have

$$F = \frac{8A^2}{gT^2} \times \frac{AC}{2CB^2 \times SP^2}$$

$$= \frac{8\pi^2 a^2 b^2}{gT^2} \times \frac{a}{2b^2 \times \varrho^2}$$

$$= \frac{4\pi^2 a^3}{gT^2} \times \frac{1}{\varrho^2} \quad \ldots\ldots\ldots\ldots \quad (a)$$

where the elements a and T are determinable by observation.

OTHERWISE.

A general expression for the force (g. 139) is

$$F = \frac{4A^2}{gT^2} \times u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

But the equation to the Ellipse gives

$$u = \frac{1}{\varrho} = \frac{1 + e\cos.\theta}{a(1 - e^2)}$$

where a is the semi-axis major and a e the eccentricity.

$$\therefore \frac{du}{d\theta} = -\frac{e\sin.\theta}{a(1 - e^2)}$$

and

$$\frac{d^2u}{d\theta^2} = -\frac{e\cos.\theta}{a(1 - e^2)}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{1}{a(1 - e^2)}$$

and

$$F = \frac{4A^2}{gT^2} \times \frac{u^2}{a(1 - e^2)}.$$

But

$$A^2 = \pi^2 a^2 b^2 = \pi^2 a^2 (a^2 - a^2 e^2)$$

$$\therefore F = \frac{4\pi^2 a^3}{gT^2} \times u^2$$

the same as before.

OTHERWISE.

Another expression is (k. 140)

$$F = \frac{4 A^2}{g T^2} \times \frac{d p}{p^3 d \varrho}.$$

Another equation to the Ellipse is also

$$\frac{1}{p^2} = \frac{2 a - \varrho}{b^2 \varrho} = \frac{2 a}{b^2 \varrho} - \frac{1}{b^2}$$

$$\therefore \frac{d p}{p^3 d \varrho} = \frac{a}{b^2 \varrho^2}$$

$$\therefore F = \frac{4 A^2}{g T^2} \times \frac{a}{b^2 \varrho^2}$$

$$= \frac{4 \pi^2 a^2 b^2}{g T^2} \times \frac{a}{b^2 \varrho^2} = \frac{4 \pi^2 a^3}{g T^2} \times \frac{1}{\varrho^2}.$$

151. Prop. XII. The same order of the proportions, which are also lettered in the same manner, as in the case of the ellipse is preserved here.

Moreover the equations to the Hyperbola are

$$\varrho = \frac{a (e^2 - 1)}{1 + e \cos. \theta}$$

and

$$p^2 = \frac{b^2 \varrho}{\varrho + 2 a}$$

which will give the same values of F as before excepting that it becomes *negative* and thereby indicates the force to be *repulsive.*

152. Prop. XIII. By Conics

$$4 S P . P v = Q v^2 = Q x^2 \text{ ultimately.}$$

But

$$P v = P x = Q R.$$

$$\therefore 4 S P . Q R : Q x^2 :: 1 : 1$$

and

$$Q x^2 : Q T^2 :: S P^2 : S N^2$$

$$:: S P : S A$$

$$\therefore 4 S P . Q R : Q T^2 :: S P : S A$$

$$\therefore \frac{Q R}{Q T^2} = \frac{1}{4 S A} = \frac{1}{L}$$

L being the latus rectum.

$$\therefore F \propto \frac{Q R}{Q T^2 \times S P^2} \propto \frac{1}{S P^2}.$$

or

$$F = \frac{8 a^2}{g} \times \frac{Q R}{S P^2 \times Q T^2} \text{ (b. 137)}$$

$$= \frac{8\,a^2}{g\,L} \times \frac{1}{S\,P^2} \text{ or } \frac{2\,P^2\,V^2}{g\,L} \times \frac{1}{S\,P^2}$$

a being the area described by the radius-vector in a second, or P the perpendicular upon the tangent and V the corresponding velocity.

OTHERWISE.

In the parabola we have

$$u = \frac{1}{\varrho} = \frac{2}{L}(1 + \cos.\theta) = \frac{2}{L} + \frac{2}{L}\cos.\theta \quad .$$

and

$$\frac{1}{p^2} = \frac{4}{L} \times \frac{1}{\varrho}$$

which give

$$\frac{d^2 u}{d\,\theta^2} + u = \frac{2}{L}$$

and

$$\frac{d\,p}{p^3\,d\,\varrho} = \frac{2}{L} \times \frac{1}{\varrho^3}$$

and these give, when substituted in

$$F = \frac{P^2\,V^2}{g} \cdot u^2 \left(\frac{d^2 u}{d\,\theta^2} + u\right)$$

or

$$= \frac{P^2\,V^2}{g} \cdot \frac{d\,p}{p^3\,d\,\varrho}$$

the same result, viz.

$$F = \frac{2\,P^2\,V^2}{g\,L} \times \frac{1}{\varrho^2} \quad \text{. . . . .} \quad \text{(b)}$$

Newton observes that the two latter propositions may easily be deduced from PROP. XI.

In that we have found (Art. 150)

$$F = \frac{4\,A^2}{g\,T^2} \times \frac{a}{b^2\,\varrho^2}$$

$$= \frac{P^2\,V^2}{g} \times \frac{a}{b^2\,\varrho^2}.$$

Now when the section becomes an Hyperbola the force must be repulsive the trajectory being convex towards the force, and the expression remains the same.

Again by the property of the ellipse

$$b^2 : \frac{L^2}{4} : a \times a : \frac{L}{4}\left(2a - \frac{L}{4}\right)$$

which gives

$$\frac{a}{b^2} = \frac{2}{L} - \frac{1}{4a}$$

and if c be the eccentricity

$$b^2 = a^2 - c^2 = (a + c) \times (a - c)$$

$$\therefore \frac{a}{(a + c) \times (a - c)} = \frac{2}{L} - \frac{1}{4a}.$$

Now when the ellipse becomes a parabola a and c are infinite, a — c is *finite*, and a + c is of the same order of *infinites* as a. Consequently $\frac{a}{b^2}$ is *finite*, and equating like quantities, we have

$$\frac{a}{b^2} = \frac{2}{L},$$

which being substituted above gives

$$F = \frac{2P^2V^2}{gL} \times \frac{1}{\varrho^2}$$

the same as before.

Again, let the Ellipse merge into a circle; then b = a and

$$F = \frac{P^2V^2}{g} \times \frac{a}{b^2\varrho^2}$$

$$= \frac{aV^2}{g} \times \frac{1}{\varrho^2}$$

$$= \frac{V^2}{g \times a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

153. Prop. XIII. Cor. 1. *For the focus, point of contact, and position of the tangent being given, a conic section can be described having at that point a given curvature.*]

For a geometrical construction see Jesuits' note, No. 268.

The elements of the Conic Section may also be thus found.

The expression for R in Art. 75 may easily be transformed to

$$R = \frac{\frac{\varrho^6}{p^3}}{\frac{\varrho^4}{p^2} + \frac{d\varrho^2}{d\theta^2} - \varrho \cdot \frac{d^2\varrho}{d\theta^2}}$$

for

$$p = \frac{\varrho^2 d\theta}{ds} = \frac{\varrho^2}{\sqrt{\left(\varrho^2 + \frac{d\varrho^2}{d\theta^2}\right)}}.$$

Now the general equation to conic sections being

$$\varrho = \frac{b^2}{a} \times \frac{1}{1 + e \cos. \theta}$$

the denominiator of the value of R is easily found to be

$$\frac{a}{b^2} \varrho^3$$

which gives

$$R = \frac{b^2}{a} \times \frac{\varrho^3}{p^3}.$$

Hence

$$\frac{b^2}{a} = \frac{p^3}{\varrho^3} \times R$$

is known.

Again, by the equation to conic sections we have

$$p^2 = \frac{b^2 \varrho}{2 a \mp \varrho}$$

which, by aid of the above, gives

$$a = \frac{\pm \varrho^3}{2 \varrho^2 - p R}.$$

And

$$b^2 = \frac{\pm p^3 R}{2 \varrho^2 - p R}.$$

Whence the construction is easy.

154. *The Curvature is given from the Centripetal Force and Velocity being given.*]

If the circle of curvature be described passing through P, Q, V, and O (P V being the chord of curvature passing through the center of force, and P O the diameter of curvature); then from the similar triangles P Q R, P V Q, we get

$$Q R = \frac{P Q^2}{P V}.$$

Also from the triangles P Q T and P S Y (S Y being the perpendicular upon the tangent) we have

$$P Q = \frac{S P \times Q T}{S Y}$$

and from P S Y, P V O,

$$P V = \frac{2 R \times S Y}{S P}$$

whence by substitution, &c.

$$\frac{QR}{QT^2 \times SP^2} = \frac{SP}{2R \times SY^2}$$

$$\therefore F = \frac{2P^2V^2}{g} \times \frac{QR}{QT^2 \times SP^2} = \frac{V^2 \times SP}{R \times SY}$$

which gives

$$R = \frac{SP}{g \cdot SY} \times \frac{V^2}{F}.$$

Hence, S P, S Y and g being given quantities, R is also given if V and F are.

155. *Two orbits which touch one another and have the same centripetal force and velocity cannot be described.*]

This is clear from the "Principle of sufficient Reason." For it is a truth axiomatic that any number of *causes* acting simultaneously under given circumstances, viz. the absolute force, law of force, velocity, direction, and distance, can produce but *one effect*. In the present case that one effect is the motion of the body in some one of the Conic Sections.

OTHERWISE.

Let the given law of force be denoted generally by $f\varrho$, where $f\varrho$ means any function; then (139)

$$F = \frac{P^2V^2}{g} \times \frac{dp}{p^3 d\varrho}$$

and since P and V are given

$$F' = \frac{P^2V^2}{g} \times \frac{dp'}{p'^3 d\varrho'}.$$

But if A be the value of F at the given distance (r) from the center to the point of contact; then

$$F : A :: f\varrho : fr$$

and

$$F' : A :: f\varrho' : fr$$

$$\therefore F = \frac{A}{fr} \times f\varrho$$

and

$$F' = \frac{A}{fr} \times f\varrho'$$

Hence

$$\frac{P^2 V^2}{g} \cdot \frac{dp}{p^3 d\varrho} = \frac{A}{fr} \times f\varrho$$

and

$$\frac{P^2 V^2}{g} \cdot \frac{dp'}{p'^3 d\varrho'} = \frac{A}{fr} \times f\varrho'$$

and integrating, we have

$$\frac{P^2 V^2 fr}{2gA} \times \left(\frac{1}{P^2} - \frac{1}{p^2}\right) = \int d\varrho f\varrho$$

and

$$\frac{P^2 V^2 fr}{2gA} \cdot \times \left(\frac{1}{P^2} - \frac{1}{p'^2}\right) = \int d\varrho' f\varrho'$$

Now $\int d\varrho f\varrho$ and $\int d\varrho' f\varrho'$ are evidently the same functions of ϱ and ϱ', which therefore assume

$$\varphi\varrho \text{ and } \varphi\varrho';$$

and adding the constant by referring to the point of contact of the two orbits, and putting

$$\frac{P^2 V^2 fr}{2gA} = M,$$

we get

$$M \times \left(\frac{1}{P^2} - \frac{1}{p^2}\right) = \varphi\varrho - \varphi r$$

$$M \times \left(\frac{1}{P^2} - \frac{1}{p'^2}\right) = \varphi\varrho' - \varphi r.$$

$$\left.\begin{aligned} \therefore \frac{1}{p^2} &= \frac{\varphi r}{M} + \frac{1}{P^2} - \frac{\varphi\varrho}{M} \\ \frac{1}{p'^2} &= \frac{\varphi r}{M} + \frac{1}{P^2} - \frac{\varphi\varrho'}{M} \end{aligned}\right\} \quad \ldots\ldots \quad (d)$$

in which equations the constants being the same, and those with which ϱ and ϱ' are also involved, the curves which are thence descriptible are *identical*. Q. e. d.

These explanations are sufficient to clear up the converse proposition contained in this corollary.

156. It may be demonstrated generally and at once as follows:

By the question

$$f\varrho = \frac{1}{\varrho^2};$$

then

$$f\,r = \frac{1}{r^2},$$

and

$$\varphi\,\varrho = \int \frac{d\,\varrho}{\varrho^2} = -\frac{1}{\varrho}$$

and substituting in (d) we have

$$\frac{1}{p^2} = -\frac{1}{r\,M} + \frac{1}{P^2} + \frac{1}{M\,\varrho}.$$

But the general equation to Conic Sections is

$$\frac{1}{p^2} = \frac{2\,a}{b^2\,\varrho} \mp \frac{1}{b^2}.$$

Whence the orbit is a Conic Section whose axes are determinable from

$$\frac{2\,a}{b^2} = \frac{1}{M} = \frac{2\,g\,A\,r^2}{P^2\,V^2}$$

and

$$\mp\frac{1}{b^2} = -\frac{1}{r\,M} + \frac{1}{P^2}$$

$$= \frac{1}{P^2} - \frac{2\,g\,A\,r}{P^2\,V^2};$$

and the section is an Ellipse, Parabola or Hyperbola according as

V^2 is $>$, or $=$ or $<$ $2\,g\,A\,r$.

Before this subject is quitted it may not be amiss by these forms also to demonstrate the converse of Prop. X, or Cor. 1, Prop. X.

Here

$$f\,\varrho = \varrho$$
$$f\,r = r$$
$$\varphi\,\varrho = \int \varrho\,d\,\varrho = \frac{\varrho^2}{2}$$
$$\varphi\,r = \frac{r^2}{2}.$$

Whence

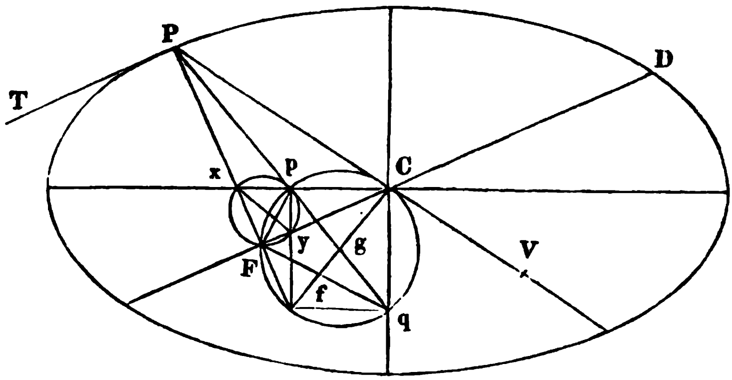
$$\frac{1}{p^2} = \frac{r^2}{2\,M} + \frac{1}{P^2} - \frac{\varrho^2}{2\,M}.$$

But in the Conic Sections referred to the center, we have

$$\frac{1}{p^2} = \frac{1}{a^2} \pm \frac{1}{b^2} \mp \frac{\varrho^2}{a^2\,b^2}$$

which shows the orbit to be an Ellipse or Hyperbola and its axes may be found as before.

In the case of the Ellipse take the following geometrical solution and construction



C, the center of force and distance C P are given. The body is projected at P with the given velocity V. Hence P V is given, (for $V^2 = \frac{g}{2} F . PV$.) Also the position of the tangent is given, ∴ position of D C is given, and $PV = \frac{2CD^2}{PC}$. Hence C D is given in magnitude. Draw P F perpendicular to C D. Produce and take P f = C D. Join C f and bisect in g. Join P g, and take g C, g f, g p, g q, all equal. Draw C p, C q. These are the positions of the major and minor axis. Also ½ major axis = P q, ½ minor axis = P p.

For from g describe a circle through C, f, p, q, and since C F f is a right ∠, it will pass through F.

$$\therefore Pp.Pq = PF.Pf = PF.CD$$

Also

$$PC^2 + Pf^2 = Pg^2 + gC^2 + Pg^2 + gf^2, \text{ (since base of } \triangle \text{ bisected in } g)$$

or

$$\begin{aligned} PC^2 + CD^2 &= Pg^2 + gq^2 + Pg^2 + gp^2 \\ &= Pq^2 - 2Pg.gq + Pp^2 + 2Pg.gp \\ &= Pq^2 + Pp^2 \end{aligned}$$

$$\left.\begin{aligned} \therefore Pp.Pq &= PF.CD \\ Pp^2 + Pq^2 &= PC^2 + CD^2 \end{aligned}\right\} \text{But a and b are determined by the same equations. } \therefore Pq = a, Pp = b.$$

Also since p and F are right angles, the circle on x y will pass through p and F, and ∠ P p x = C p q = C F q = x F p, because ∠ x F C = p F q.

∴ ∠ P p x = ∠ in alternate segment. ∴ P p is tangent.

$$Pp^2 = PF.Px \qquad \therefore PF.Px = b^2.$$

But if in the Ellipse C x be the major axis, $PF.Px = b^2$.

$\therefore$ C x is the major axis, and $\therefore$ C q is the minor axis.

$\therefore$ the Ellipse is constructed.

Prop. XIII, Cor. 2. See Jesuits' note. The case of the body's descent in a straight line to the center is here omitted by Newton, because it is possible in most laws of force, and is moreover reserved for a full discussion in Section VII.

The value of the force is however easily obtained from 140.

157. Prop. XIV. $$L = \frac{QT^2}{QR} \propto \frac{QT^2}{F}$$
$$\propto QT^2 \times SP^2 \text{ by hypoth.}$$

OTHERWISE.

By Art. 150,

$$F = \frac{4A^2}{gT^2} \times \frac{a}{b^2 \varrho^2} = \frac{8A^2}{LgT^2} \times \frac{1}{\varrho^2}$$

for the circle, ellipse, and hyperbola, and by 152.

$$F = \frac{2P^2V^2}{Lg} \times \frac{1}{\varrho^2}$$

for the parabola.

Now if μ be the value of F at distance 1, we have

$$F = \frac{\mu}{\varrho^2}.$$

Whence in the former case

$$\frac{8A^2}{gLT^2} = \mu, \text{ or } = \frac{2P^2 \times V^2}{gL} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and in the latter

$$\frac{2P^2 \times V^2}{gL} = \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

But

$$\frac{SP^2 \times QT^2}{4} : 1^2 : A^2 : T^2$$

$$\therefore \frac{A^2}{T^2} = \frac{SP^2 \times QT^2}{4} = \frac{P^2 \times V^2}{4}$$

$$\therefore SP^2 \times QT^2 = \frac{\mu g}{2} L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

158. Prop. XIV. Cor. I. By the form (a) we have

$$A\ (= \pi a b) = \sqrt{\frac{\mu g}{8}} \times \sqrt{L} \times T.$$
$$\propto T \sqrt{L}.$$

159. PROP. XV. From the preceding Art.

$$T = \sqrt{\frac{8}{\mu g}} \times \frac{\pi a b}{\sqrt{L}}.$$

But in the ellipse

$$L = \frac{2 b^2}{a}$$

$$\therefore T = \frac{2 \pi}{\sqrt{\mu g}} \times a^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

160. PROP. XVI. For explanations of the text see Jesuits' notes.

OTHERWISE.

By Art. 157 we get

$$V = \sqrt{\frac{g \mu}{2}} \times \frac{\sqrt{L}}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

for the circle, ellipse, hyperbola, and parabola.

But in the circle, $L = 2 P$.

$$\therefore V = \sqrt{g \mu} \times \frac{1}{\sqrt{P}} = \sqrt{g \mu} \times \frac{1}{\sqrt{r}} \quad . \quad . \quad . \quad (g)$$

r being its radius.

In the ellipse and hyperbola

$$L = \frac{2 b^2}{a}$$

$$\therefore V = \sqrt{g \mu} \times \frac{b}{\sqrt{a}} \times \frac{1}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

161. PROP. XVI, COR. I. By 157,

$$L = \frac{2}{g \mu} \times P^2 \times V^2.$$

162. COR. 2. $V = \sqrt{\frac{g \mu}{2}} \times \frac{\sqrt{L}}{D}$,

D being the max. or min. distance.

163. COR. 3. By Art. 160, and the preceding one,

$$V : V' :: \sqrt{\frac{g \mu}{2}} \times \frac{\sqrt{L}}{D} : \sqrt{g \mu} \times \frac{1}{\sqrt{D}}$$

$$:: \sqrt{L} : \sqrt{2 D}.$$

164. COR. 4. By Art. 160,

$$v : v' :: \sqrt{\frac{g \mu}{2}} \times \frac{\sqrt{L}}{P} : \sqrt{g \mu} \times \frac{1}{\sqrt{r}}$$

But

$$L = \frac{2\,b^2}{a},\ P = b,\ \text{and } r = a$$

$$\therefore v : v' :: \frac{b}{b\sqrt{a}} : \frac{1}{\sqrt{a}} :: 1 : 1.$$

165. Cor. 6. By the equations to the parabola, ellipse, and hyperbola, viz.

$$p^2 = \frac{L}{4} \times \varrho,\ p^2 = \frac{b^2}{2\,a - \varrho},\ \text{and } p^2 = \frac{b^2}{2\,a + \varrho}$$

the Cor. is manifest.

166. Cor. 7. By Art. 160 we have

$$v^2 : v'^2 :: \frac{1}{2} \cdot \frac{L}{P^2} : \frac{1}{r}$$

which by aid of the above equations to the curves proves the Cor.

OTHERWISE.

By Art. 140 generally for all curves

$$v^2 = g\,F \times \frac{P\,V}{2}.$$

But generally

$$P\,V = \frac{2\,p\,d\,\varrho}{d\,p}$$

and in the circle

$$P\,V = 2\,\varrho\ (\text{rad.} = \varrho)$$

$$\therefore v^2 : v'^2 :: \frac{p}{\varrho} : \frac{d\,p}{d\,\varrho}.$$

An analogy which will give the comparison between v and v′ for any curve whose equation is given.

167. Cor. 9. By Cor. 8,

$$v : v' :: \frac{L}{2} : p$$

and

$$v' : v'' :: \sqrt{\varrho} : \sqrt{\frac{L}{2}}$$

∴ ex equo

$$v : v'' :: \sqrt{\frac{\varrho\,L}{2}} : p.$$

168. PROP. XVII. The "*absolute quantity of the force*" must be known, viz. the value of μ, or else the *actual* value of V in the assumed orbit will not be determinable; i. e.

$$L : L' :: P^2 V^2 : P'^2 V'^2$$

will not give L'.

It must be observed that it has already been shown (Cor. 1, Prop. XIII) that the orbit is a conic section.

See Jesuits' notes, and also Art. 153 of this Commentary.

169. PROP. XVII, COR. 3. The two motions being compounded, the position of the tangent to the new orbit will thence be given and therefore the perpendicular upon it from the center. Also the new velocity. Whence, as in Prop. XVII, the new orbit may be constructed.

OTHERWISE.

Let the velocity be augmented by the impulse m times.

Now, if μ be the force at the distance 1, and P and V the perpendicular and velocity at distance (R) of projection, by 156 the general equation to the new orbit is such that its semi-axes are

$$a = \frac{R}{2 - m^2}, \text{ or } = \frac{R}{m^2 - 2}$$

and

$$b^2 = \frac{m^2 P}{2 - m^2}, \text{ or } \frac{m^2 P}{m^2 - 2}$$

according as the orbit is an ellipse or hyperbola. Moreover it also thence appears that when $m^2 = 2$, the orbit is a parabola, and that the equations corresponding to these cases are

$$p^2 = \frac{m^2 P}{2 - m^2} \times \frac{\rho}{2 \times \frac{R}{2 - m^2} - \rho}$$

or

$$= \frac{m^2 P}{m^2 - 2} \times \frac{\rho}{2 \times \frac{R}{m^2 - 2} + \rho}$$

or

$$= P \times \rho.$$

DEDUCTIONS AND ADDITIONS

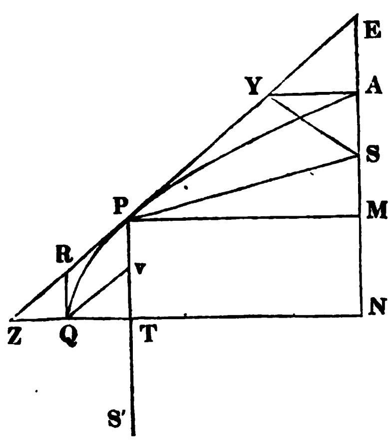
TO

SECTIONS II AND III.

170. In the parabola the force acting in lines parallel to the axis, required F.

$4\,SP.QR : QT^2 :: Qv^2 : QT^2 :: YE^2 : YA^2 :: SE : SA :: SP : SA$

$\therefore \frac{QR}{QT^2} = \frac{1}{4\,SA}$, and $S'P$ is constant, $\therefore$ F is constant.



Let u be the velocity resolved parallel to P M then since the force acts perpendicular to P M, u at any point must be same as at A, $\therefore$ if P Q be the velocity in the curve, $QT = u =$ constant quantity, and $a = \frac{S'P.QT}{2}$ $\frac{S'P.u}{2}$.

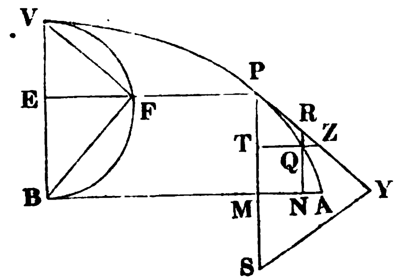
$$\therefore F = \frac{8\,a^2.QR}{g\,S'P^2.QT^2} = \frac{2\,u^2}{g\,L} \text{ (see 157)}$$

which avoids the consideration of S P being infinite; and

$$u^2 = 2\,g\,F.\frac{L}{4}$$

$\therefore$ body must fall through $\frac{L}{4}$ to acquire the velocity at vertex, which agrees with Mechanics. (At any point $V = u\sqrt{\frac{SP}{SA}}$.)

171. In the cycloid required the force when acting parallel to the axis.



$$R P^2 : Q T^2 :: Z P^2 : Z T^2 :: V F^2 : E F^2 :: V B : B E$$

and since the chord of curvature (C. c) $= 4 P M$, $R P^2 = 4 P M . R Q$,

$$\therefore 4 P M . R Q : Q T^2 :: V B : (B E =) P M$$

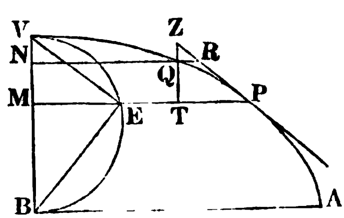
$$\therefore \frac{Q R}{Q T^2} = \frac{V B}{4 P M^2},$$

$$\therefore F \propto \frac{1}{P M^2} \text{ (since S P constant)}$$

$$F = \frac{8 a^2 . Q R}{g . S P^2 . Q T^2} = \frac{u^2 . V B}{2 g . P M^2}, \text{ if } u = \text{velocity parallel to A B.}$$

$$\left(\text{At any point } v = u . \sqrt{\frac{B V}{P M}} .\right)$$

172. In the cycloid the force is parallel to the base



$$R P^2 : Q T^2 :: Z P^2 : Z T^2 :: V E^2 : V M^2 :: V B : V M$$

and since $C . c = 4 E M$

$$R P^2 = 4 E M . R Q,$$

$$\therefore 4 E M . R Q : Q T^2 :: V B : V M,$$

$$\therefore \frac{Q R}{Q T^2} = \frac{V B}{4 E M . V M} \propto \frac{1}{E M . V M} .$$

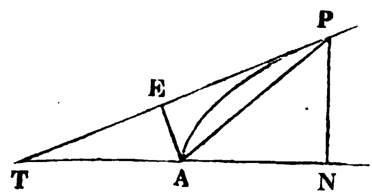
$$\text{If } V M = y, \; F = \frac{u^2 r}{g y \sqrt{2 r y - y^2}} \left(r = \frac{V B}{2}\right)$$

u = velocity parallel to V B.

$$\left(F = \frac{8 a^2 Q R}{g . S P^2 Q T^2} = \frac{2 u^2 . Q R}{g . Q T^2} = \frac{u^2 . V B}{2 g . E M . V M}.\right)$$

$$\left(\text{At any point } v = u . \sqrt{\frac{B V}{V M}}.\right)$$

173. *Find* F *in a parabola tending to the vertex.*



$$T P : P N :: T A : A E$$

or

$$\sqrt{4 x^2 + y^2} : y :: x : \frac{y x}{\sqrt{4 x^2 + y^2}} = p, (A E),$$

$$\therefore \frac{1}{p^2} = \frac{4 x^2 + a x}{a x^3} = \frac{4 x + a}{a x^2}$$

$$\therefore -\frac{2 d p}{p^3} = \frac{4 d x . a x^2 - 2 a x d x (4 x + a)}{a^2 x^4}$$

$$= -\frac{4 x^2 + 2 a x}{a x^4} . d x = -\frac{2}{a} . \frac{2 x + a}{x^3} . d x,$$

$$\therefore \frac{d p}{p^3} = \frac{2 x + a}{a x^3} . d x.$$

Also

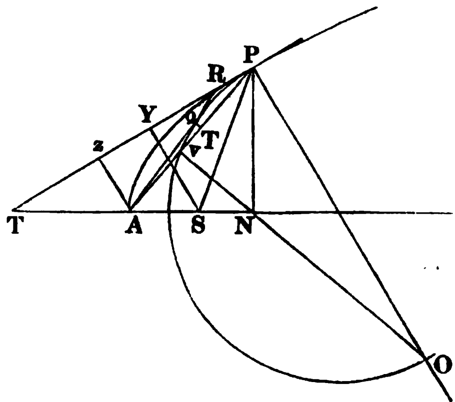
$$\varrho = \sqrt{x^2 + y^2},$$

$$\therefore d \varrho = \frac{x d x + y d y}{\sqrt{x^2 + y^2}} = \frac{x d x + \frac{a d x}{2}}{\sqrt{x^2 + a x}}$$

$$\therefore \frac{d p}{p^3 d \varrho} = \frac{2 x + a}{a x^3} . \frac{2 \sqrt{x^2 + a x}}{2 x + a} = \frac{2 \sqrt{x^2 + a x}}{a x^3}$$

$$\therefore F \propto \frac{A P}{A N^3}.$$

174. *Geometrically.* Let P Q O be the circle of curvature,



$$P\,v \text{ (C. c through the vertex of the parabola)} = \frac{P\,O\,.\,A\,z}{A\,P}$$

$$\therefore \frac{P\,Q^2}{Q\,R} = \frac{P\,O\,.\,A\,z}{A\,P}$$

but

$$\frac{P\,Q^2}{Q\,T^2} = \frac{A\,P^2}{A\,z^2}$$

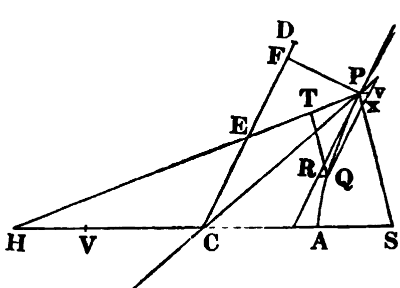
$$\therefore \frac{Q\,R}{Q\,T^2} = \frac{A\,P^3}{P\,O\,.\,A\,z^3}$$

$$\therefore F = \frac{8\,a^2\,.\,Q\,R}{g\,.\,A\,P^2\,.\,Q\,T^2} = \frac{8\,a^2\,.\,A\,P}{g\,.\,P\,O\,.\,A\,z^3}$$

but

$$P\,O\,.\,A\,z^3 = 2\,A\,S\,.\,\frac{S\,P^3}{S\,Y^3}\,.\,\frac{S\,Y^3\,A\,T^3}{S\,P^3} = 2\,A\,S\,.\,A\,N^3$$

$$\therefore F = \frac{4\,a^2\,.\,A\,P}{g\,.\,A\,S\,.\,A\,N^3}.$$



175. If the centripetal be changed into a repelling force, and the body revolve in the opposite hyperbola, $F \propto \frac{1}{H\,P^2}$.

The body is projected in direction P R; R Q is the deflection from the Tangent due to repelling force H P, find the force.

$$L.Px : L.Pv :: Px : Pv :: PE : PC :: AC : PC$$

$$L.Pv : Pv.vG :: L : 2PC$$

$$Pv.vG : Qv^2 :: PC^2 : CD^2$$

$$Qv^2 : Qx^2 :: 1 : 1$$

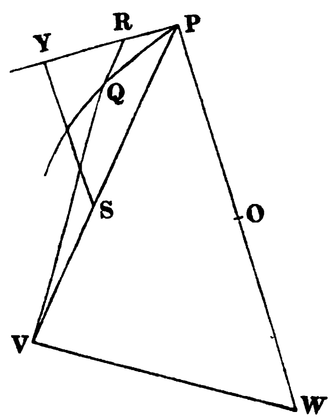
$$Qx^2 : QT^2 :: PE^2 : PF^2 :: AC^2 : PF^2 :: CD^2 : BC^2$$

$$\therefore L.Px : QT^2 :: AC.L.PC^2.CD^2 : 2PC^2.CD^2.BC^2$$

$$:: L : \frac{2BC^2}{AC} :: 1 : 1$$

$$\therefore L = \frac{QT^2}{QR}$$

$$\therefore F = \frac{8a^2.QR}{g.HP^2.QT^2} = \frac{8a^2}{g.L.HP^2} \propto \frac{1}{HP^2}.$$



176. In any Conic Section the chord of curvature $= \frac{L.SP^2}{SY^2}$

for

$$PV = \frac{QP^2}{QR} = \frac{QT^2.SP^2}{QR.SY^2} = \frac{L.SP^2}{SY^2}.$$

177. Radius of curvature $= \frac{L}{2}\cdot\frac{SP^3}{SY^3}$;

for

$$PW = \frac{PV.SP}{SY} = \frac{L.SP^3}{SY^3}.$$

178. Hence in any curve $F = \frac{8a^2}{g.SY^2.PV}$

$$= \frac{8a^2}{g.SY^2.\frac{2R.SY}{SP}} = \frac{4a^2.SP}{g.SY^3.R}.$$ see Art. 74.

179. Hence in Conic Sections

$$F = \frac{8a^2}{g.SY^2.PV} = \frac{8a^2}{g.SY^2.\frac{L.SP^2}{SY^2}} = \frac{8a^2}{g.L.SP^2} \propto \frac{1}{SP^2}.$$

180. If the chord of curvature be proved $= \frac{L.SP^2}{SY^2}$ independently of he proof that $\frac{QT^2}{QR} = L$, this general proof of the variation of force in tonic sections might supersede Newton's; otherwise not.

181. *A body attached to a string, whose length* $= b$, *is whirled round so as to describe a circle whose center is the fixed extremity of the string parallel to the horizon in* T″; *required the ratio of the tension to the weight.*

Gravity $= 1$, $\therefore$ v of the revolving body $= \sqrt{gFb}$, if b be the length of the string;

$$\therefore F\ (= \text{centripetal force} = \text{tension}) = \frac{V^2}{gb}\ (131)$$

and

$$T = \frac{\text{circumference}}{V} = \frac{2\pi b}{\sqrt{gFb}} = 2\pi\frac{\sqrt{b}}{\sqrt{gF}}$$

$$\therefore F = \frac{\pi^2 4b}{gT^2}$$

$\therefore F : \text{Gravity} :: \frac{4\pi^2 b}{gT^2} : 1$, or Tension : weight $:: 4\pi^2 b : gT^2$.

If Tension $= 3$ weight; required T.

$$4\pi^2 b : gT^2 :: 3 : 1,$$

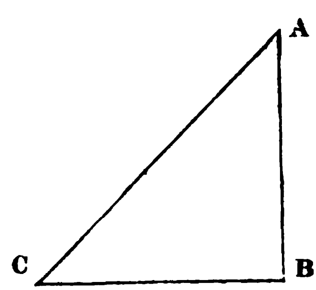
$$\therefore T^2 = \frac{4\pi^2 b}{3g}.$$

If T be given, and the tension $=3$ weight, required the length of the string.

$$T^2 = \frac{4\pi^2 b}{3g},$$

$$\therefore b = \frac{3T^2 g}{4\pi^2}.$$

182. *If a body suspended by a string from any point describe a circle, the string describes a cone; required the time of one revolution or of one oscillation.*



Let $AC = l$, $BC = b$,

The body is kept at rest by 3 forces, gravity in the direction of A B, tension in the direction C A, and the centripetal force in the direction C B.

As before, centripetal force $= \dfrac{4\pi^2 b}{g T^2}$,

and centripetal force : gravity : b : $\sqrt{l^2 - b^2}$, (from $\triangle$) : : $\dfrac{4\pi^2 b}{g T^2}$: 1

$$\therefore T^2 = \frac{4\pi^2 \sqrt{l^2 - b^2}}{g}$$

$$\therefore T = 2\pi \sqrt{\frac{\sqrt{l^2 - b^2}}{g}} = \text{a constant quantity if } \sqrt{l^2 - b^2}$$

be given.

∴ the time of oscillation is the same for all conical pendulums having a common altitude.

183. v *in the Ellipse at the perihelion* : v *in the circle* e. d. : : n : 1, *find the major axis, excentricity, and compare its* T *with that in the circle, and find the limits of* n.

Let S A = c,

v in the Ellipse : that in the circle e. d. : : $\sqrt{HP} : \sqrt{AC}$
: : $\sqrt{HA} : \sqrt{AC}$ in this case
: : n : 1 by supposition,

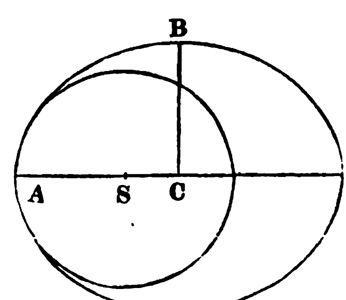
$$\therefore 2\,AC - AS = n^2\,AC,$$

$$\therefore AC = \frac{c}{2 - n^2}.$$

$$\text{Excentricity} = AC - AS = \frac{c}{2 - n^2} - c = \frac{c\,n^2 - c}{2 - n^2}.$$

$$T : T \text{ in the circle} :: AC^{\frac{3}{2}} : AS^{\frac{3}{2}} :: \frac{c^{\frac{3}{2}}}{(2 - n^2)^{\frac{3}{2}}} : c^{\frac{3}{2}} :: 1 : (2 - n^2)^{\frac{3}{2}}$$

Also n must be $< \sqrt{2}$,
for if n $= \sqrt{2}$, the orbit is a parabola
if n $> \sqrt{2}$, the orbit is an hyperbola.



184. *Suppose* $\frac{1}{4}$ *of the quantity of matter of* ⊕ *to be taken away. How much would* T *of* ☽ *be increased, and what the excentricity of her new orbit? the* ☽*'s present orbit being considered circular.*

At any point A her direction is perpendicular to S A,

∴ if the force be altered at any point A, her v in the new orbit will

$=$ her v in the circle, since $v = \frac{2\alpha}{SY}$, and $SY = SA$, and α is the same at A.

Let $AS = c$, PV at $A = L$, and $F = \frac{2V^2}{g\,PV} \propto \frac{1}{PV}$ in this case,

$$\therefore 3 : 4 :: 2c \ (= L \text{ in the circle}) : \frac{2b^2}{a} \ (= L \text{ in the ellipse})$$

$$\therefore 4c = \frac{3b^2}{a} = \frac{3(a^2 - \overline{a-c}^2)}{a} = \frac{3(2ac - c^2)}{a} = 6c - \frac{3c^2}{a}$$

$$\therefore \frac{3c^2}{a} = 2c,$$

$$\therefore a = \frac{3c}{2}.$$

And T in the circle : T′ in the ellipse $:: \frac{c^{\frac{3}{2}}}{\sqrt{4}} : \frac{\left(\frac{3c}{2}\right)^{\frac{3}{2}}}{\sqrt{3}}$

$$:: \frac{\sqrt{3}}{\sqrt{4}} : \left(\frac{3}{2}\right)^{\frac{3}{2}} :: \frac{1}{\sqrt{2}} : \frac{3}{2}$$

$$:: \sqrt{2} : 3.$$

And the excentricity $= a - c = \frac{3}{2}c - c = \frac{c}{2}$.

185. *What quantity must be destroyed that ☽'s T may be doubled, and what the excentricity of her new orbit?*

Let F of ⊕ : f (new force) :: n : 1

$$\therefore v = \sqrt{\frac{g}{2} F \cdot PV}, \text{ and v is given,}$$

$$\therefore F \propto \frac{1}{PV},$$

$$\therefore n : 1 :: \frac{2b^2}{a} : 2c :: \frac{a^2 - \overline{a-c}^2}{a} : c :: \frac{2ac - c^2}{a} : c :: 2a - c : a^2$$

$$\therefore na = 2a - c,$$

$$\therefore a = \frac{c}{2-n}.$$

Also T in the circle : T in the ellipse :: 1 : 2

$$:: \frac{c^{\frac{3}{2}}}{\sqrt{n}} : \frac{c^{\frac{3}{2}}}{(2-n)^{\frac{3}{2}}}$$

$$:: (2-n)^{\frac{3}{2}} : n^{\frac{1}{2}}$$

$\therefore 1 : 4 :: (2-n)^3 : n \ \therefore n = 4(2-n)^3$, whence n.

And the excentricity

$$= a - c = \frac{c}{2 - n} - c = \frac{c - (2\,c - n\,c)}{2 - n} = \frac{c\,(n - 1)}{2 - n}$$

186. *What quantity must be destroyed that ☽'s orbit may become a parabola?*

$$L = 4\,c,$$

$$\therefore F : f :: 4\,c : 2\,c :: 2 : 1,$$

$\therefore \frac{1}{2}$ the force must be destroyed.

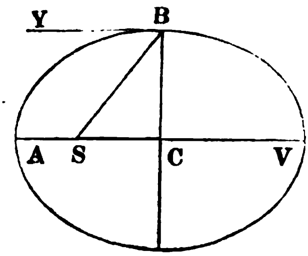
187. $F \propto \frac{1}{D^2}$, *a body is projected at given* D, v = v *in the circle,* ∠ *with* S B = 45°, *find axis major, excentricity, and* T.

Since v = v in the circle, ∴ the body is projected from B,

and ∠ S B Y = 45°;

∴ ∠ S B C, or B S C = 45°,

$$\therefore S\,C = S\,B.\cos.45^\circ = \frac{S\,B}{\sqrt{2}}.$$

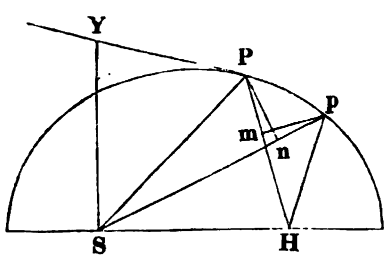


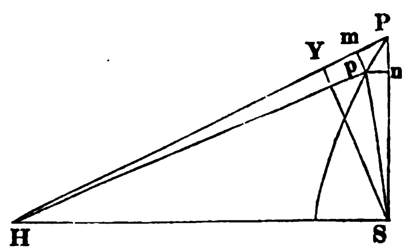
But

$$S\,B = D = \frac{\text{axis major}}{2},$$

∴ axis major and excentricity are found.

And T may be found from Art. 159.





188. *Prove that the angular* v *round* H : *that round* S :: S P : H P.

This is called Seth Ward's Hypothesis.

In the ellipse. Let P m, p n, be perpendicular to S p, H P,

∴ p m = Increment of S P = Decrement of H P = P n

∴ triangles P m p, P n p, are equal,

$$\therefore P\,m = p\,n, \text{ and angular v} \propto \frac{1}{\text{distance}}$$

189. Similarly in the hyperbola.

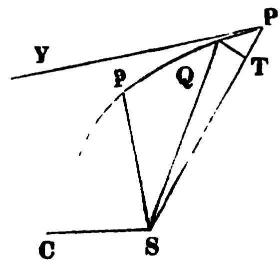
$$\text{Angular v of } S\,P : \text{angular v of } S\,Y :: P\,V : 2\,S\,P :: \frac{2\,C\,D^2}{A\,C} : 2\,S\,P$$

$$:: \frac{C\,D^2}{S\,P} : A\,C$$

$$:: H\,P : A\,C.$$

190. *Compare the times of falling to the center of the logarithmic spiral from different points.*

The times are as the areas.



$$d . \text{area} = \frac{\varrho^2 d\theta}{2}, \ (\theta = \angle CSP), \text{ for } d . \text{area} = \frac{QT . SP}{2}.$$

$$\text{Also } \frac{QT}{TP} = \frac{\varrho . d\theta}{d\varrho} = \tan. \angle YPT = \tan. \alpha, \ (\alpha \text{ being constant}) = a$$

$$\therefore d\theta = \frac{a . d\varrho}{\varrho},$$

$$\therefore \frac{\varrho^2 d\theta}{2} = \frac{a . \varrho . d\varrho}{2},$$

$$\therefore \text{area} = \frac{a . \varrho^2}{4} \propto \varrho^2, \ (\text{for when } \varrho = 0, \ \text{area} = 0, \therefore \text{Cor.} = 0)$$

$\therefore$ if P, p, be points given,

T from P to center : t from p to center : : $SP^2 : Sp^2$.

191. *Compare* v *in a logarithmic spiral with that in a circle,* e. d.

$$F = \frac{2V^2}{g . PV} \ (140)$$

$\therefore$ if F be given, $V \propto \sqrt{PV}$,

$\therefore$ v in spiral : v in the circle : : $\sqrt{PV \text{ in spiral}} : \sqrt{2SP}$: : 1 : 1.

192. *Compare* T *in a logarithmic spiral with that in a circle,* e. d.

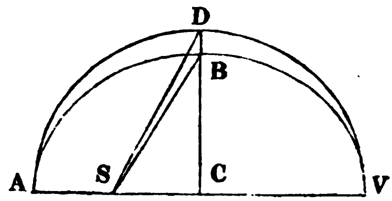
$$T \text{ in spiral} = \frac{\text{whole area}}{\text{area in } 1''} = \frac{a\varrho^2}{4 . \frac{v . SY}{2}} = \frac{a\varrho^2}{2v . \varrho . \sin. \alpha}$$

$$T' \text{ in circle} = \frac{\text{whole area}}{\text{area in } 1''} = \frac{\pi\varrho^2}{\frac{v . SY}{2}} = \frac{2\pi\varrho^2}{v . \varrho} = \frac{2\pi\varrho}{v}$$

$$\therefore T : T' : : \frac{a\varrho^2}{2v . \varrho . \sin. \alpha} : \frac{2\pi\varrho}{v} : : \frac{a}{2\sin. \alpha} : 2\pi : : a : 4\pi . \sin. \alpha.$$

$$: : \tan. \alpha : 4\pi . \sin. \alpha : : 1 : 4\pi \cos. \alpha.$$

192. *In the Ellipse compare the time from the mean distance to the Aphelion, with the time from the mean distance to the Perihelion. Also given the Excentricity, to find the difference of the times, and conversely.*



$$A\,D\,V \text{ is } \frac{\text{Circle}}{2} \text{ described on } A\,V.$$

T of passing through Aphelion : t through Perihelion

$$:: S\,B\,V : S\,B\,A$$
$$:: S\,D\,V : S\,D\,A$$
$$:: C\,D\,V + \frac{D\,C\,.\,S\,C}{2} : C\,D\,V - \frac{D\,C\,.\,S\,C}{2}.$$

Let $Q =$ quadrant C D V,

$$\therefore T : t :: Q + \frac{a\,.\,a\,e}{2} : Q - \frac{a\,.\,a\,e}{2}$$
$$\therefore (T + t =)\,P : T - t :: 2\,Q : a\,.\,a\,e$$
$$\therefore T - t = \frac{P\ \ a\,.\,a\,e}{2\,Q}$$

whence $T - t$, or, if $T - t$ be given, a e may be found.

193. *If the perihelion distance of a comet in a parabola* $= 64$, $\oplus$*'s mean distance* $= 100$, *compare its velocity at the extremity of* L *with* $\oplus$*'s velocity at mean distance.*

Since $\oplus$ moves in an ellipse, v at the mean distance = that in the circle e . d . and v in the parabola at the extremity of L

$$: v \text{ in the circle rad. } 2\,S\,A :: \sqrt{2} : 1$$

v in the circle rad. 2 S A

$$: v \text{ in the circle rad. } A\,C :: \sqrt{A\,C} : \sqrt{S\,A}$$

$\therefore$ v in the parabola at L

$$: v \text{ in the ellipse at } B :: \sqrt{2\,.\,A\,C} : \sqrt{S\,A\,.\,2}$$
$$:: 10\sqrt{2} : 8\sqrt{2}$$
$$:: 5 : 4$$

194. *What is the difference between* L *of a parabola and ellipse, having the same* $<^{st}$ *distance* $= 1$, *and axis major of the ellipse* $= 300$? *Compare the* v *at the extremity of* L *and* $<^{st}$ *distances.*

In the parabola $L = 4\,A\,S = 4$.

In the ellipse $L' = \frac{2\,BC^2}{AC} = \frac{2}{300}(AC^2 - \overline{AC - SA}^2)$

$$= \frac{1}{150}(2\,AC\,.\,AS - AS^2) = \frac{600-1}{150}.$$

$$\therefore L - L' = 4 - \left(4 - \frac{1}{150}\right) = \frac{1}{150}.$$

v in the parabola at A : v in the circle rad. S A : : $\sqrt{2} : 1$

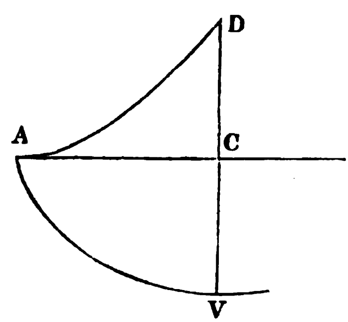
v in the circle rad. S A : v in the ellipse e. d. : : $\sqrt{AC} : \sqrt{HP}$

: : $\sqrt{AC} : \sqrt{2\,AC - SA}$: : $\sqrt{150} : \sqrt{299}$

$\therefore$ v in the parabola at A : v in the ellipse e. d. : : $\sqrt{300} : \sqrt{299}$.

Similarly compare v^2. at the extremity of Lat. R.

195. *Suppose a body to oscillate in a whole cycloidal arc, compare the tension of the string at the lowest point with the weight of the body.*



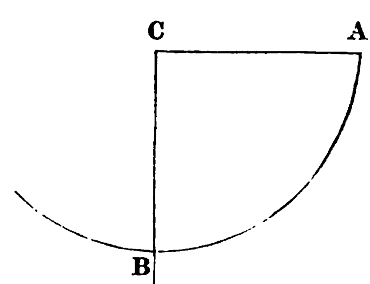
The tension of the string arises from two causes, the weight of the body, and the centrifugal force. At V we may consider the body revolving in the circular arc rad. D V, $\therefore$ the centrifugal = centripetal force. Now the velocity at V = that down C V by the force of grav.

= that with which the body revolves in the circle rad. 2 C V.

$\therefore$ grav. : centrifugal force : : 1 : 1,

$\therefore$ tension : grav. : : 2 : 1

196. *Suppose the body to oscillate through the quadrant* A B, *compare the tension at* B *with the weight.*



At B the string will be in the direction of gravity; $\therefore$ the whole weight will stretch the string; $\therefore$ the tension will = centrifugal force + weight. Now the centrifugal force = centripetal force with which the body would revolve in the circle e. d.

And v in the circle = $\sqrt{2\,g\,.\,F\,.\,\frac{R}{2}}$

$$\therefore F = \frac{v^2}{g R} = \frac{v^2}{g C B} \text{ in this case,}$$

also v′ at B from grav. $= \sqrt{2 g . C B}$, grav. = 1.

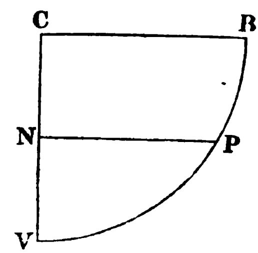
$$\therefore \text{grav.} = 1 = \frac{v'^2}{2 g C B}$$

$$\therefore F : \text{grav.} : : \frac{v^2}{2gCB} : \frac{v'^2}{g C B} : : 2 : 1,$$

since v = v′.

$$\therefore \text{tension} : \text{grav.} : : 3 : 1.$$

197. *A body vibrates in a circular arc from the center* C; *through what arc must it vibrate so that at the lowest point the tension of the string* = 2 × *weight?*



v from grav. = v d. N V, (if P be the point required) v′ of revolution in the circle = v d. $\frac{C V}{2}$.

$$\therefore \text{centrifugal force} : \text{grav.} : : v : v' : : \sqrt{\frac{C V}{2}} : \sqrt{N V}$$

$$\therefore \text{centrifugal force} + \text{grav.} (= \text{tension}) : \text{grav.} : : \sqrt{\frac{C V}{2}} + \sqrt{N V} : \sqrt{N V}$$

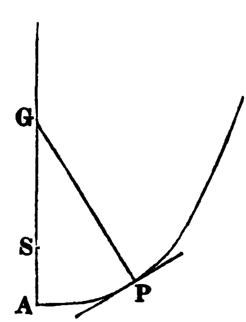
: : 2 : 1 by supposition.

$$\therefore \sqrt{\frac{C V}{2}} + \sqrt{N V} = 2 \sqrt{N V}$$

$$\therefore \sqrt{\frac{C V}{2}} = \sqrt{N V},$$

$$\therefore N V = \frac{C V}{2}.$$

198. *There is a hollow vessel in form of an inverted paraboloid down which a body descends, the pressure at lowest point* = n . *weight, find from what point it must descend.*



At any point P, the body is in the same situation as if suspended from G, P G being normal, and revolving in the circle whose rad. G P. Now P G = $\sqrt{4 A S . S P}$, ∴ at A, P G =

$\sqrt{4 AS^2} = 2 AS$. Also v^2 at A with which the body revolves $= 2g . F . \frac{2AS}{2}$.

$$\therefore \text{centrifugal force} = \frac{v^2}{2 g AS}$$

$$\text{and grav.} = \frac{v^2}{2 g h}, \text{ if } h = \text{height fallen from.}$$

But the whole pressure arises from grav. + centrifugal force, and $=$ n . grav.

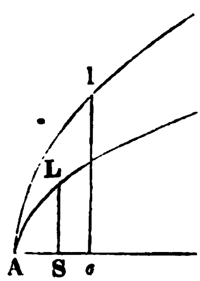
$$\therefore \text{centrifugal force} + \text{grav.} : \text{grav.} :: n : 1$$

or

$$\frac{1}{AS} + \frac{1}{h} : \frac{1}{h} :: n : 1,$$

$$\therefore \frac{1}{AS} : \frac{1}{h} :: n - 1 : 1;$$

$$\therefore h = \overline{n - 1} . AS.$$



199. *Compare the time* (T′) *in which a body describes* 90° *of anomaly in a parabola with* T *in the circle rad.* = S A.

Time through A L : 1 :: area A S L : a in 1″

$$\therefore T = \frac{\frac{2}{3} AS . SL}{a} = \frac{4 AS^2}{3 a}$$

T in the circle rad. S A : 1 :: whole circle : a′ in 1″

$$\therefore T = \frac{\pi AS^2}{a'}$$

$$\therefore T' : T :: \frac{4}{3 a} : \frac{\pi}{a'}$$

and

$$a : a' :: \sqrt{L} : \sqrt{2 AS} :: \sqrt{4 AS} : \sqrt{2 AS} :: \sqrt{2} : 1$$

$$\therefore T' : T :: \frac{4}{3 \sqrt{2}} : \pi :: 2 \sqrt{2} : 3 \pi.$$

Compare the time of describing 90° *in the parabola* A L *with that in the parabola* A l, (fig. same.)

$$t : T \text{ in the circle rad. } SA :: 4 : 3 \sqrt{2} . \pi$$

T in the circle S A : T′ in the circle rad. σ A :: $SA^{\frac{3}{2}} : \sigma A^{\frac{3}{2}}$ (since $T^2 \propto R^3$)

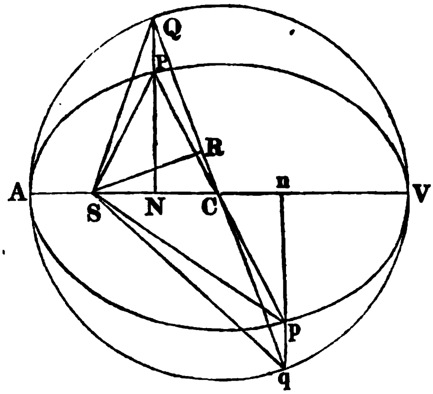
$$T' : t' \text{ through } l A :: 3 \sqrt{2} . \pi : 4$$

$$\therefore t \text{ through } SA : t' \text{ through } \sigma A :: SA^{\frac{3}{2}} : \sigma A^{\frac{3}{2}}.$$

See Sect. VI. Prop. XXX.

200. *Draw the diameter* P p *such that the time through* P V p : *time through* p A P : : n : 1, *force* $\propto \frac{1}{SP^2}$.

Describe the circle on A V.



Let t = time through P V p, and T the periodic time

$$\therefore \frac{t}{T} = \frac{n}{n+1} = \frac{PVpS}{\text{ellipse}} = \frac{QVqS}{\text{circle}} = \frac{\text{circle} + \triangle QqS}{\text{circle}}$$

$$= \frac{\frac{\text{circle}}{2} + \frac{SR \cdot 2CQ}{2}}{\text{circle}}$$

$$= \frac{\frac{\pi a^2}{2} + a\,e \,.\, \sin. u \,.\, a}{\pi a^2}, \text{ (u = excentric anomaly)}$$

$$= \frac{\frac{\pi}{2} + e \,.\, \sin. u}{\pi}$$

$$\therefore n\pi = \overline{n+1} \,.\, \left(\frac{\pi}{2} + e \sin. u\right)$$

$$= n\frac{\pi}{2} + \frac{\pi}{2} + \overline{n+1} \,.\, e \sin. u$$

$$\therefore \sin. u = \frac{n-1}{n+1} \cdot \frac{\pi}{2e},$$

which determines u, &c.

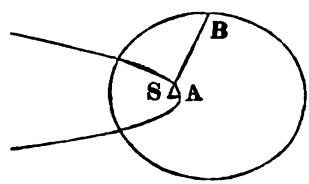
201. *The Moon revolves round the Earth in* 30 *days, the mean distance from the Earth* = 240,000 *miles. Jupiter's Moon revolves in* ½ *day, the mean distance from Jupiter* = 240,000 *miles. Compare the absolute forces of Jupiter and the Earth.*

$T \propto \frac{A^{\frac{3}{2}}}{\sqrt{\mu}}$, A being the major axis of the ellipse,

$\therefore$ If A be given, $\mu \propto \frac{1}{T^2}$;

$$\therefore \frac{\text{Mass of Jupiter}}{\text{Mass of the Earth}} = \frac{T^2 \text{ of the Earth's Moon}}{T'^2 \text{ of Jupiter's Moon}} = \frac{30^2}{\frac{1}{4^2}} = \frac{14,400}{1}.$$

202. *A Comet at perihelion is 400 times as near to the Sun as the Earth at its mean distance. Compare their velocities at those points.*



$$\frac{\text{Velocity}^2 \text{ of the Comet}}{\text{Velocity}^2 \text{ of the Earth}} = \frac{F \cdot 4\,A\,S}{F' \cdot 2\,B\,S} = \frac{F}{F'} \cdot \frac{4}{2 \cdot 400} = \frac{F}{F'} \cdot \frac{1}{200}$$

$$= \frac{\overline{400}^2}{1^2} \cdot \frac{1}{200} = 800$$

$$\therefore \frac{V}{v} = \frac{\sqrt{2 \cdot 20}}{1} = \frac{30}{1} \text{ nearly.}$$

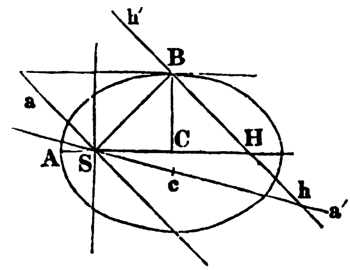
203. *Compare the Masses of the Sun and Earth, having the mean distance of the Earth from the Sun = 400, the distance of the Moon from the Earth, and Earth's* P^{d}. = 13. *the Moon's* P^{d}.

$$T^2 \propto \frac{a^3}{\mu},$$

$$\therefore \mu \propto \frac{a}{T^2},$$

$$\therefore \frac{\text{Mass of the Sun}}{\text{Mass of the Earth}} = \frac{400^3}{1^3} \cdot \frac{1^2}{13^2} = \frac{64,000,000}{169} = 400,000 \text{ nearly.}$$

204. *If the force* $\propto \frac{1}{x^2} - \frac{a}{x^3}$, *where* x *is the distance from the center of force, it will be centripetal whilst* $\frac{1}{x^2} > \frac{a}{x^3}$, *or* $x > a$; *there will be a point of contrary flexure in the orbit when* $\frac{1}{x^2} = \frac{a}{x^3}$, *or* $x = a$, *and afterwards when* $x < a$, *the force will be repulsive, and the curve change its direction.*



205. *The body revolving in an ellipse, at* B *the force becomes* n *times as great. Find the new orbit, and under what values of* n *it will be a parabola, ellipse, or hyperbola.*

S being one focus since the force $\propto \frac{1}{\text{distance}^2}$, the other focus must lie in B H produced both ways, since S B, H B, make equal angles with the tangent. $V^2 = \frac{g}{2} F . P V = \frac{g}{2} F . 2 A C$ in the original ellipse, or $= \frac{g}{2} n F . P V$ in the new orbit.

$$\therefore 2 A C = n . P V = n . \frac{2 S B . h B}{\frac{S B + h B}{2}},$$

$$\therefore (S B + h B) A C = 2 n . S B . h B,$$

$$\therefore A C^2 + h B . A C = 2 n A C . h B,$$

$$\therefore h B = \frac{A C}{2 n - 1}.$$

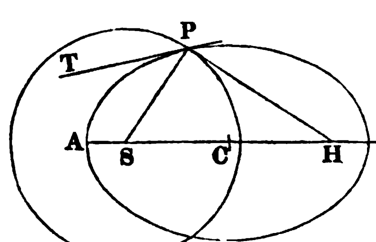
If $2 n - 1 = 0$, or $n = \frac{1}{2}$, the orbit is a parabola; if $n > \frac{1}{2}$, the orbit is an ellipse; if $n < \frac{1}{2}$, the orbit is an hyperbola.

Let S C in the original ellipse be given = B C,

$\therefore$ S B H = right angle, and S B or A C = B h . cot. B S h whence the direction of a a′, the new major axis; also

$$a a' = S B + B h, \text{ and } S c = \frac{S h}{2} = \frac{\sqrt{B h^2 - S B^2}}{2}.$$

If the orbit in the parabola a a′ be parallel to B h, and L . R = 2 S B, since S B h = right angle.



206. *Suppose a Comet in its orbit to impel the Earth from a circular orbit in a direction making an acute angle with the Earth's distance from the Sun, the velocity after impact being to the velocity before* : : $\sqrt{3} : \sqrt{2}$. *Find the alteration in the length of the year.*

Since $\sqrt{3} : \sqrt{2} <$ ratio than $\sqrt{2} : 1$, $\therefore$ the new orbit will be an ellipse.

$$\frac{V^2}{v^2} = \frac{3}{2} = \frac{PV}{2SP} = \frac{2SP.HP}{AC.2SP} = \frac{HP}{AC}$$

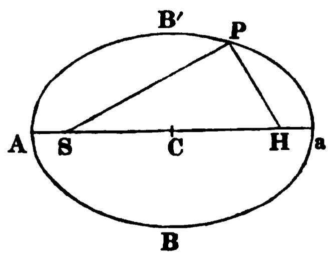
$$= \frac{2AC - SP}{AC}$$

$$\therefore 3AC = 4AC - 2SP$$

$$\therefore 2SP = AC$$

$$\therefore \frac{\text{T in ellipse}}{\text{T}' \text{ in circle}} = \frac{2^{\frac{3}{2}} SP^{\frac{3}{2}}}{SP^{\frac{3}{2}}} = \frac{8}{3} \text{ nearly.}$$

207. *A body revolves in an ellipse, at any given point the force becomes diminished by $\frac{1}{n}$th part. Find the new orbit.*



$$v^2 \propto F.PV$$

$$\therefore \text{in this case } PV \propto \frac{1}{F},$$

$$\therefore \frac{\text{P V in ellipse}}{\text{pv in new orbit}} = \frac{1 - \frac{1}{n}}{1} = \frac{n-1}{n}$$

But

$$\frac{v^2 \text{ in conic section}}{v^2 \text{ in circle e. d.}} = \frac{pv}{2SP} = \frac{\frac{n}{n-1}.PV}{2SP} \text{ at P}$$

$$= \frac{n}{n-1}.\frac{HP}{AC}$$

$$\therefore \text{if } \frac{n}{n-1}.HP = AC, \text{ the new orbit is a Circle}$$
$$\text{———} = 2AC, \text{———} \text{ Parabola}$$
$$\text{———} < 2AC, \text{———} \text{ Ellipse}$$
$$\text{———} > 2AC, \text{———} \text{ Hyperbola}$$

If $\frac{n}{n-1} = 2$, or $n = 2$, then when the orbit is a circle or an ellipse, P must be between a, B; when the orbit is a parabola, P must be at B; when the orbit is an hyperbola, P must be between B, A.

208. *If the curvature and inclination of the tangent to the radius be the same at two points in the curve, the forces at those points are inversely as the radii[2].*

$$F = \frac{8\,a^2}{g.SY^2.PV} = \frac{8\,a^2}{g.SY.SP.R} = \frac{8\,a^2}{g.\sin.\theta\,SP^2.R} \propto \frac{1}{SP^2}.$$

This applies to the extremities of major axis in an ellipse (or circle) in the center of force in the axis.

209. *Required the angular velocity of ϱ.*

By 46, θ being the traced-angle,

$$\omega = \frac{d\,\theta}{d\,t}.$$

But by Prop. I. or Art. 124,

$$d\,t : T :: d\,A : A$$

$$:: \frac{\varrho^2 d\,\theta}{2} : A \ (113)$$

$$\therefore \omega = \frac{d\,\theta}{d\,t} = \frac{2\,A}{T} \times \frac{1}{\varrho^2}$$

or

$$= \frac{P \times V}{\varrho^2} \quad \ldots\ldots\ldots\ldots \quad (a)$$

210. *Required the Centrifugal Force (φ) in any orbit.*

When the revolving body is at any distance ϱ from the center of force, the Centrifugal Force, which arises from its inertia or tendency to persevere in the direction of the tangent (most authors erroneously attribute this force to the angular motion, see Vince's Flux. p. 283) is clearly the same as it would be were the body with the same Centripetal Force revolving in a circle whose radius is ϱ. Moreover, since in a circle the body is always at the same distance from the center, the Centrifugal Force must always be equal to the Centripetal Force.

But in the circle

$$Q\,T^2 = Q\,R \times 2\,S\,P$$

and $\therefore$ by 137 we have

$$F = \frac{8\,A^2}{g\,T^2} \times \frac{1}{2\,S\,P^3} = \frac{4\,A^2}{g\,T^2} \times \frac{1}{\varrho^3}$$

or

$$= \frac{P^2\,V^2}{g} \times \frac{1}{\varrho^3}$$

P and V belonging to the orbit.

Hence then

$$\varphi = \frac{P^2 V^2}{g} \times \frac{1}{\varrho^3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

Hence also and by 209,

$$\varphi = \frac{\omega^2 \times \varrho}{g} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (b)$$

And 139,

$$F : \varphi :: \frac{d\,p}{p^3} : \frac{d\,\varrho}{\varrho^3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (c)$$

211. *Required the angular velocity of the perpendicular upon the tangent.*

If two consecutive points in the curve be taken; tangents, perpendiculars and the circle of curvature be described as in Art. 74, it will readily appear that the incremental angle ($d\,\psi$) described by p = that described by the radius of curvature. It will also be seen that

$$d\,\theta : d\,\psi \left(:: \frac{Q\,T}{\varrho} : \frac{P\,Q}{R} \right) :: \frac{p}{\varrho} : \frac{\varrho}{R}$$

But from similar triangles

$$P\,V : 2\,R :: p : \varrho.$$

$$\therefore d\,\theta : d\,\psi :: P\,V : 2\,\varrho$$

P V being the chord of curvature.

Hence

$$\varpi \left(= \frac{d\,\psi}{d\,t} \right) = \frac{d\,\theta}{d\,t} \times \frac{2\,\varrho}{P\,V} \text{ (see 209)}$$

$$= \omega \times \frac{2\,\varrho}{P\,V} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (d)$$

or

$$= \frac{2\,P \times V}{\varrho \times P\,V} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (e)$$

or

$$= \frac{P \times V}{p\,\varrho} \times \frac{d\,p}{d\,\varrho} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (f)$$

Ex. 1. In the circle P V = $2\,\varrho$; whence

$$\varpi = \frac{P \times V}{\varrho^2} = \omega.$$

Ex. 2. In the other Conic Sections, we have

$$p^2 = \frac{b^2\,\varrho}{2\,a \mp \varrho}$$

which gives by taking the logarithms

$$2 \, l \, p = l \, b^2 + l \, \varrho - l \, (2 \, a \mp \varrho)$$

and (17 a.)

$$\frac{2 \, d \, p}{p} = \frac{d \, \varrho}{\varrho} \pm \frac{d \, \varrho}{2 \, a \mp \varrho} = \frac{2 \, a \, d \, \varrho}{\varrho \, (2 \, a \mp \varrho)}$$

whence

$$\varpi = \frac{a \, P \times V}{\varrho^2 \, (2 \, a \mp \varrho)}.$$

212. *Required the Paracentric Velocity in an orbit.*

It readily appears from the fig. that

$$d \, s : d \, \varrho :: \varrho : \sqrt{\varrho^2 - p^2}.$$

∴ If u denote the velocity towards the center, we have

$$u \left(= \frac{d \, \varrho}{d \, t}\right) = \frac{d \, s}{d \, t} \times \frac{\sqrt{\varrho^2 - p^2}}{\varrho}$$

$$= \frac{P \times V}{p} \times \frac{\sqrt{\varrho^2 - p^2}}{\varrho} \quad (125)$$

$$= P \, V \times \sqrt{\left(\frac{1}{p^2} - \frac{1}{\varrho^2}\right)} \quad . \quad . \quad . \quad . \quad (g)$$

or

$$= \frac{2 \, A}{T} \times \sqrt{\left(\frac{1}{p^2} - \frac{1}{\varrho^2}\right)} \quad . \quad . \quad . \quad . \quad (h)$$

Also since

$$\frac{1}{p^2} = \frac{\varrho^2 \, d \, \theta^2 + d \, \varrho^2}{\varrho^4 \, d \, \theta^2}$$

$$u = P \, V \times \frac{d \, \varrho}{\varrho^2 \, d \, \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

213. *To find where in an orbit the Paracentric Velocity is a maximum.*

From the equation to the curve substitute in the expression (212. g) for p^2, then put $d \, u = 0$, and the resulting value of ϱ will give the position required.

Thus in the ellipse

$$p^2 = \frac{b^2 \, \varrho}{2 \, a - \varrho}$$

and

$$u^2 = P^2 \, V^2 \times \left(\frac{2 \, a - \varrho}{b^2 \, \varrho} - \frac{1}{\varrho^2}\right) = \max.$$

$$\therefore \frac{2 \, a}{b^2 \, \varrho} - \frac{1}{\varrho^2} - \frac{1}{b^2} = \max.$$

$$\therefore -\frac{2\,a\,d\,\varrho}{b^2\,\varrho^2} + \frac{2\,d\,\varrho}{\varrho^3} = 0$$

and

$$\varrho = \frac{b^2}{a} = \frac{\text{Latus-Rectum}}{2}$$

or the point required is the extremity of the Latus-Rectum.

OTHERWISE.

Generally, It neither increases nor decreases when $F = \varphi$. Hence when $u = \max.$ (see 210)

$$\frac{d\,p}{p^3} = \frac{d\,\varrho}{\varrho^3}.$$

which is also got from putting

$$d\,(u^2) = 0$$

in the expression 212. h.

214. *To find where the angular velocity increases fastest.*

By Art. 209 and 125,

$$\frac{d\,\omega}{d\,t} = 2\,P\,V \times \frac{d\,\varrho}{\varrho^3} \times \frac{P\,V}{\varrho^2\,d\,\theta} = \frac{2\,P^2\,V^2}{\varrho^4} \times \frac{d\,\varrho}{\varrho\,d\,\theta},$$

$$\therefore \frac{d\,\varrho}{\varrho^5\,d\,\theta} = \max. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But from similar triangles

$$p : \surd\,(\varrho^2 - p^2) :: Q\,T : P\,T :: \varrho\,d\,\theta : d\,\varrho$$

$$\therefore \frac{d\,\omega}{d\,t} = \frac{2\,P^2\,V^2}{p\,\varrho^4} \times \surd\,(\varrho^2 - p^2) = \max.$$

$$\therefore \frac{\varrho^2 - p^2}{p^2\,\varrho^8} = \frac{1}{p^2\,\varrho^6} - \frac{1}{\varrho^8} = \max. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

either of which equations, by aid of that to the curve, will give the point required.

Ex. In the ellipse

$$p^2 = \frac{b^2\,\varrho}{2\,a - \varrho}$$

$$\therefore \frac{2\,a - \varrho}{b^2\,\varrho^7} - \frac{1}{\varrho^8} = \max. = m$$

and $\frac{d\,m}{d\,\varrho} = 0$ gives

$$\varrho^2 - \frac{7}{3}\,a\,\varrho = -\frac{4}{3}\,b^2$$

which gives

$$\varrho = \frac{7}{6}\,a \pm \frac{1}{6}\ \sqrt{(49\,a^2 - 48\,b^2)}$$

for the maxima or minima positions.

If the equation

$$\varrho = \frac{b^2}{a} \times \frac{1}{1 + e\cos.\theta}$$

and the first form be used, we have

$$\frac{d\,\varrho}{d\,\theta} = \frac{a\,e}{b^2} \times \varrho^2 \sin.\theta$$

and

$$\frac{\sin.\theta}{\varrho^3} = \text{max.} = m.$$

Whence and from d m = 0, we get finally

$$\cos.\theta = -\frac{1}{8\,e} \pm \sqrt{\left(\frac{1}{64\,e^2} + \frac{3}{4}\right)}.$$

215. *To find where the Linear Velocity increases fastest.*

Here

$$\frac{d\,v}{d\,t} = \text{max.}$$

But (125)

$$v = \frac{P \times V}{p}$$

and

$$d\,t = \frac{\varrho^2\,d\,\theta}{P \times V} = \frac{\varrho}{P \times V} \times \frac{p\,d\,\varrho}{\sqrt{\varrho^2 - p^2}}$$

$$\therefore \frac{d\,v}{d\,t} = \frac{P^2 V^2 \sqrt{(\varrho^2 - p^2)}}{\varrho} \times \frac{d\,p}{p^3\,d\,\varrho}$$

$$= g\,F \times \frac{\sqrt{(\varrho^2 - p^2)}}{\varrho}.$$

$$\left.\begin{array}{l} \therefore \dfrac{\sqrt{(\varrho^2 - p^2)}}{\varrho} \times \dfrac{d\,p}{p^3\,d\,\varrho} \\ \text{or} \\ F \times \dfrac{\sqrt{(\varrho^2 - p^2)}}{\varrho} \end{array}\right\} = \text{max.} = m.$$

and

$$d\,m = 0$$

will give the point required.

Thus in the ellipse

$$F \propto \frac{1}{\varrho^2}$$

$$\therefore \frac{\varrho^2 - p^2}{\varrho^6} = \frac{1}{\varrho^4} - \frac{b^2}{2 a \varrho^5 - \varrho^6} = \text{max.}$$

$$\therefore \frac{d m}{d \varrho} = -\frac{4}{\varrho^5} + \frac{10 a b^2 \varrho^4 - 6 b^2 \varrho^5}{(2 a - \varrho)^2 \varrho^{10}} = 0$$

which gives

$$\varrho^3 + 4 a \varrho^2 - \frac{8 a^2 + 3 b^2}{2} \varrho + \frac{5}{2} a b^2 = 0,$$

whence the maxima and minima positions.

In the case of the parabola, a is indefinitely great and the equation becomes

$$4 a^2 \varrho - \frac{5}{2} a b^2 = 0$$

$$\therefore \varrho = \frac{5}{8} \times \frac{b^2}{a} = \frac{5}{16} \times \text{Latus-Rectum.}$$

Many other problems respecting velocities, &c. might be here added. But instead of dwelling longer upon such matters, which are rather curious than useful, and at best only calculated to exercise the student, I shall refer him to my Solutions of the Cambridge Problems, where he will find a great number of them as well as of problems of great and essential importance.

SECTION IV.

216. PROP. XVIII. If the two points P, p, be given, then circles whose centers are P, p, and radii A B $\mp$ S P, A B $\mp$ S p, might be described intersecting in H.

If the positions of two tangents T R, t r be given, then perpendiculars S T, S t must be let fall and doubled, and from V and v with radii each = A B, circles must be described intersecting in H.

Having thus in either of the three cases determined the other focus H, the *ellipse* may be described *mechanically*, by taking a thread = A B in length, fixing its ends in S and H, and running the pen all round so as to stretch the string.

This proposition may thus be demonstrated analytically.

1st. Let the focus S, the tangent T R, and the point P be given in position; and the axis-major be given in length, viz. 2 a. Then the perpendicular S T (= p), and the radius-vector S P (= ϱ) are known.

But the equation to Conic Sections is

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho}$$

whence b is found.

Also the distance (2 c) between the foci is got by making p = ϱ, thence finding ϱ and therefore $c = a \mp \varrho$.

This gives the other focus; and the two foci being known, and the axis-major, the curve is easily constructed.

217. 2d. Let two tangents T R, t r, and the focus S be given in position. Then making S the origin of coordinates, the equations to the trajectory are

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho}, \text{ and } \varrho = \frac{b^2}{a} \cdot \frac{1}{1 + e \cos. (\theta - \alpha)} \quad . \quad . \quad . \quad (a)$$

α being the inclination of the axis-major to that of the abscissæ.

Now calling the angles which the tangents make with the axis of the abscissæ T and T', by 31 we have

$$\tan. T = \frac{dy}{dx}.$$

But

$$x = \varrho \cos. \theta, \ y = \varrho \sin. \theta$$

whence

$$\tan. T = \frac{d\varrho \sin. \theta + \varrho d\theta \cos. \theta}{d\varrho \cos. \theta - \varrho d\theta \sin. \theta}$$

$$= \frac{\frac{\varrho d}{\varrho d\theta} \tan. \theta + 1}{\frac{d\varrho}{\varrho d\theta} - \tan. \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Also from equations (a) we easily get

$$\frac{d\varrho}{\varrho d\theta} = \frac{ae}{b^2} \cdot \varrho \sin. (\theta - \alpha) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos. (\theta - \alpha) = \frac{b^2 - a\varrho}{ae\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\sin. (\theta - \alpha) = \frac{b}{ae\varrho} \times \sqrt{(2a\varrho - \varrho^2 - b^2)} \quad . \quad . \quad (3)$$

and

$$\varrho = \frac{2ap^2}{p^2 \pm b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and putting

$$R = \sqrt{(2 a \rho - \rho^2 - b^2)} \quad . \quad . \quad . \quad . \quad (5)$$

we have

$$\frac{R}{b^2 - a \rho} = \tan.(\theta - \alpha) = \frac{\tan.\theta - \tan.\alpha}{1 + \tan.\alpha . \tan.\theta} \quad . \quad (6)$$

which gives tan. θ in terms of a, b, ρ, and tan. α.

Hence by successive substitutions by means of these several expressions tan. T may be found in terms of a, b, p, tan. α, all of which are given except b and tan. α. Let, therefore,

$$\tan. T = f(a, p, b, \tan.\alpha).$$

In like manner we also get

$$\tan. T' = f(a, p', b, \tan.\alpha)$$

p′ belonging to the tangent whose inclination to the axis is T.

From these two equations b and tan. α may be found, which give $c = \sqrt{a^2 - b^2}$ and α, or the distance between the foci and the position of the axis-major; which being known the Trajectory is easily constructed.

218. 3d. Let the focus and two points in the curve be given in position, &c.

Then the corresponding radii ρ, ρ', and traced angles θ, θ', in the equations

$$\rho = \frac{\pm a (1 - e^2)}{1 + e \cos.(\theta - \alpha)}$$

$$\rho' = \frac{\pm a (1 - e^2)}{1 + e \cos.(\theta' - \alpha)}$$

are given; and by the formula

$$\cos.(\theta - \alpha) = \cos.\theta . \cos.\alpha + \sin.\theta \sin.\alpha$$

2 a e and α or the distance between the foci and the position of the axis-major may hence be found.

This is much less concise than Newton's geometrical method. But it may still be useful to students to know both of them.

219. PROP. XIX. To make this clearer we will state the three cases separately.

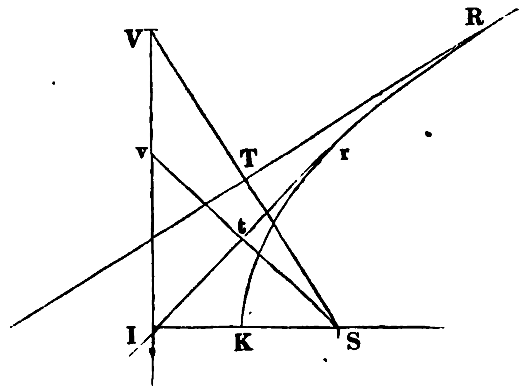
Case 1. Let a point P and tangent T R be given.

Then the figure in the text being taken, we double the perpendicular S T, describe the circle F G, and draw F I touching the circle in F and passing through V. But this last step is thus effected. Join V P, suppose it to cut the circle in M (not shown in the fig.), and take

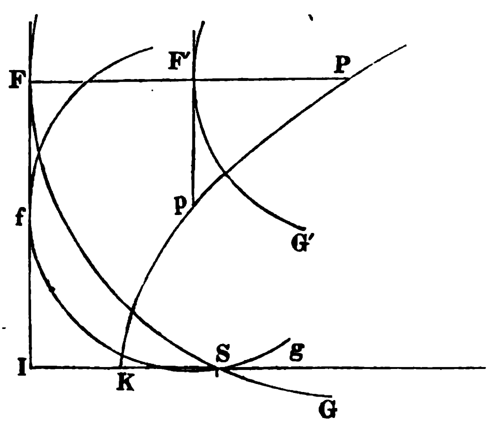
$$V F^2 = V M \times (V P + P M).$$

The rest is easy.

Case 2. Let two tangents be given. Then V and v being determined the locus of them is the directrix. Whence the rest is plain.



Case 3. Let two points (P, p) be given. Describe from P and p the circles F G, f g intersecting in the focus S. Then draw F f a common tangent to them, &c.



But this is done by describing from P with a radius = S P — S p, a circle F′ G′, by drawing from p the tangent p F′ as in the other case (or by describing a semicircle upon P p, so as to intersect F′ G′ in F′) by producing P F′ to F, and drawing F f parallel to F′ p.

See my Solutions of the Cambridge Problems, vol. I. Geometry, where tangencies are fully treated.

These three cases may easily be deduced analytically from the general solution above; or in the same way may more simply be done at once, from the equations

$$p^2 = \frac{L}{4}\,\varrho,\ \varrho = \frac{L}{2} \times \frac{1}{1 + \cos.\,(\theta - \alpha)}.$$

220. PROP. XX. Case 1. *Given in species*] means the same as "similar" in the 5th LEMMA.

Since the Trajectory is given in species, &c.] From p. 36 it seems that the ratio of the axes 2 a, 2 b is given in similar ellipses, and thence the same is easily shown of hyperbolas. Hence, since

$$c^2 = a^2 \mp b^2$$

2 c being the distance between the foci, if $\frac{b}{a} = m$, a given quantity, we have

$$\frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} = \sqrt{(1 \mp m^2)} = e,$$

which is also given.

With the centers B, C, &c.]

The common tangent L K is drawn as in 219.

Cases 2. 3. See Jesuits' Notes.

OTHERWISE.

221. Case 1. Let the two points B, C and the focus S be given. Then

$$\left.\begin{aligned} \varrho &= \frac{\pm a\,(1 - e^2)}{1 + e \cos.\,(\theta - \alpha)} \\ \varrho' &= \frac{\pm a\,(1 - e^2)}{1 + e \cos.\,(\theta' - \alpha)} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

α being the inclination of the axis of abscissæ to the axis major.

But since the trajectory is given in species

$$e = \frac{c}{a} \text{ is known,}$$

and in equations (1), ϱ, θ; ϱ', θ', are given.

Hence, therefore, by the form

$$\cos.\,(\theta - \alpha) = \cos.\,\theta\,.\,\cos.\,\alpha + \sin.\,\theta\,\sin.\,\alpha,$$

a and α, or the semi-axis-major and its position are found;

$$\text{also } c = a\,e \text{ is known;}$$

which gives the construction.

Case 2. By proceeding as in 220, in which expressions (e) will be known, both a, a e, and α may be found.

Case 3. In this case

$$p^2 = \frac{b^2 \varrho}{2a \mp \varrho} = \frac{a^2 \times (1 - e^2) \varrho}{2a \mp \varrho}$$

will give a. Hence c = a e is known and

$$\varrho = \frac{\pm a (1 - e^2)}{1 + e \cos. (\theta - \alpha)}$$

gives α.

Case 4. Since the trajectory to be described must be similar to a given one whose a′ and c′ are given,

$$\therefore e = \frac{c}{a} = \frac{c'}{a'}$$

is known (217).

Also ϱ and θ belonging to the given point are known.

Hence we have

$$\varrho = \frac{\pm a . (1 - e^2)}{1 + e \cos. (\theta - \alpha)}$$

And by means of the condition of touching the given line, another equation involving a, α may be found (see 217) which with the former will give a and α.

222. Scholium to Prop. XXI.

Given three points in the Trajectory and the focus to construct it.

ANOTHER SOLUTION.

Let the coordinates to the three points be ϱ, θ; ϱ', θ'; ϱ'', θ'', and α the angle between the major axis and that of the abscissæ. Then

$$\left.\begin{aligned} \varrho &= \frac{\pm a . (1 - e^2)}{1 + e \cos. (\theta - \alpha)} \\ \varrho' &= \frac{\pm a (1 - e^2)}{1 + e \cos. (\theta' - \alpha)} \\ \varrho'' &= \frac{\pm a (1 - e^2)}{1 + e \cos. (\theta'' - \alpha)} \end{aligned}\right\} \quad (A)$$

and eliminating $\pm a (1 - e^2)$ we get

$$\left.\begin{aligned} \varrho - \varrho &= e . \cos. (\theta' - \alpha) - e \cos. (\theta - \alpha) \\ \varrho - \varrho' &= e . \cos. (\theta'' - \alpha) - e \cos. (\theta - \alpha) \end{aligned}\right\} \quad (B)$$

from which eliminating e, there results

$$\frac{\rho - \rho'}{\rho' . \cos. (\theta' - \alpha) - \rho \cos. (\theta - \alpha)} = \frac{\rho - \rho''}{\rho'' \cos. (\theta'' - \alpha) - \rho \cos. (\theta - \alpha)}$$

Hence by the formula

$$\cos. (P - Q) = \cos. P . \cos. Q + \sin. P . \sin. Q$$

$$\tan. \alpha = \frac{(\rho - \rho')\rho'' \cos. \theta'' - (\rho - \rho'') \rho' \cos. \theta' + \rho (\rho' - \rho'') \cos. \theta}{(\rho - \rho') \rho'' \sin. \theta'' - (\rho - \rho'') \rho' \sin. \theta' + \rho (\rho' - \rho'') \sin. \theta}$$

which gives α.

Hence by means of equations (B) e will be known; and then by substitution in eq. (A), a is known.

SECTION V.

The preliminary LEMMAS of this section are rendered sufficiently intelligible by the Commentary of the Jesuits P.P. Le Seur, &c.

Moreover we shall be brief in our comments upon it (as we have been upon the former section) for the reason that at Cambridge, the focus of mathematical learning, the students scarcely even touch upon these subjects, but pass at once from the third to the sixth section.

223. PROP. XXII.

This proposition may be analytically resolved as follows:

The general equation to a conic section is that of two dimensions (see Wood's Alg. Part IV.) viz.

$$y^2 + A x y + B x^2 + C y + D x + E = 0$$

in which if A, B, C, D, E were given the curve could be constructed.

Now since five points are given by the question, let their coordinates be

$$\alpha, \beta; \ \alpha_1, \beta_1; \ \alpha_2, \beta_2; \ \alpha_3, \beta_3; \ \alpha_4, \beta_4.$$

These being substituted for x, y, in the above equation will give us five simple equations, involving the five unknown quantities A, B, C, D, E, which may therefore be easily determined; and then the trajectory is easily constructed by the ordinary rules (see Wood's Alg. Lacroix's Diff. Cal. &c.)

224. PROP. XXIII. The analytical determination of the trajectory from these conditions is also easy.

Let

$$\alpha, \beta; \ \alpha_1, \beta_1; \ \alpha_2, \beta_2; \ \alpha_3, \beta_3$$

be the coordinates of the given point. Also let the tangent given in position be determinable from the equation

$$y' = m x' + n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

in which m, n are given.

Then first substituting the above given values of the coordinates in

$$y^2 + A x y + B x^2 + C y + D x + E = 0 \quad . \quad . \quad . \quad (b)$$

we get *four* simple equations involving the five unknown quantities A, B, C, D, E; and secondly since the inclination of the curve to the axis of abscissæ is the same at the point of contact as that of the tangent,

$$\frac{d y}{d x} = \frac{d y'}{d x'}$$

$$y = y'$$

$$x = x'$$

$$\therefore \frac{A y + 2 B x + D}{2 y + A x + C} = -m$$

and substituting in this and the general equation for y its value

$$y' = m x + n$$

we have

$$\frac{A (m x + n) + 2 B x + D}{2 (m x + n) + A x + C} = -m$$

and

$$(m x + n)^2 + A x (m x + n) + B x^2 + C (m x + n) + D x + E = 0,$$

from the former of which

$$x = -\frac{n A + m C + D}{2 (m^2 + m A + B)}$$

and from the latter

$$x = -\frac{1}{2 (m^2 + m A + B)} \times (n A + m C + D + 2 m n$$

$$\mp \sqrt{\{(n A + m C + D + 2 m n)^2 - (n^2 + n C + E)(m^2 + m A + B)\}}$$

and equating these and reducing the result we get

$$4 m^2 n^2 = (n A + m C + D + 2 m n)^2 - (n^2 + n C + E)(m^2 + m A + B)$$

and this again reduces to

$$n^2 A^2 + m^2 C^2 + D^2 + m n A C + 2 n A D$$
$$+ 2 m C D - n B C - m A E - B E + 3 m n^2 A$$
$$+ 3 n m^2 C + 4 m n D - n^2 B - m^2 E - n^2 m^2 = 0$$

which is a fifth equation involving A, B, C, D, E.

From these five equations let the five unknown quantities be determined, and then construct eq. (b) by the customary methods.

225. PROP. XXIV.

OTHERWISE.

Let

$$\alpha, \beta;\ \alpha', \beta';\ \alpha'', \beta''$$

be the coordinates of the three given points, and

$$y' = m x' + n$$
$$y'' = m' x'' + n'$$

the equations to the two tangents. Then substituting in the general equation for Conic Sections these pairs of values of x, y, we get three *simple* equations involving the unknown coefficients A, B, C, D, E; and from the conditions of contact, viz.

$$\left.\begin{array}{l}\frac{dy}{dx} = \frac{dy'}{dx'} = m\\ y = y'\\ x = x'\end{array}\right\}, \quad \left.\begin{array}{l}\frac{dy}{dx} = \frac{dy''}{dx''} = m'\\ y = y''\\ x = x''\end{array}\right\}$$

We also have two other equations (see 224) involving the same five unknowns, whence by the usual methods they may be found, and then the trajectory constructed.

226. PROP. XXV.

Proceeding as in the last two articles, we shall get two simple equations and three quadratics involving A, B, C, D, E, from whence to find them and construct the trajectory.

227. PROP. XXVI.

In this case we shall have one simple equation and four quadratics to find A, B, C, D, E, with, and wherewith to describe the orbit.

228. PROP. XXVII.

In the last case of the five tangents we shall have five quadratics wherewith to determine the coefficients of the general equation, and to construct.

SECTION VI.

229. Prop. XXX.

OTHERWISE.

After a body has moved t″ from the vertex of the parabola, let it be required to find its position.

If A be the area described in that time by the radius vector, and P, V the perpendicular or the tangent and velocity at any point, by 124 and 125 we have

$$A = \frac{c}{2} \times t = \frac{PV}{2} \times t$$

and by 157,

$$PV = \sqrt{\frac{g\mu L}{2}}$$

L being the latus-rectum.

$$\therefore A = \frac{\sqrt{g\mu L}}{2\sqrt{2}} \times t.$$

But

$$ASP = AOP - SOP = \tfrac{2}{3} AO \times PO - \tfrac{1}{2} SO \times PO$$
$$= \tfrac{2}{3} xy - \tfrac{1}{2} \cdot (x - r) y$$

where r = A S, &c. (see 21) and

$$y^2 = 4rx$$
$$\therefore A = \sqrt{\frac{g\mu r}{2}} \times t = \frac{1}{6} xy + \frac{r}{2} \cdot y = \frac{y^3}{24r} + \frac{r}{2} y$$
$$\therefore y^3 + 12r^2 y = 12rt\sqrt{g\mu r}$$

by the resolution of which y may be found and therefore the position of P.

OTHERWISE.

230. By 46 and 125,

$$dt = \frac{ds}{v} = \frac{p\,ds}{c}.$$

Also

$$ds = -\frac{\varrho\, d\varrho}{c\sqrt{(\varrho^2 - p^2)}}$$

$$\therefore dt = \frac{p\rho d\rho}{c\sqrt{(\rho^2 - p^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

which is an expression of general use in determining the time in terms of the radius vector, &c.

In the parabola

$$p^2 = r\rho,$$

whence

$$dt = \frac{\sqrt{r}}{c} \times \frac{\rho d\rho}{\sqrt{(\rho - r)}}$$

and integrating *by parts*

$$t = \frac{2\sqrt{r}}{c}\rho\sqrt{(\rho - r)} - \frac{2\sqrt{r}}{c}\int d\rho\sqrt{(\rho - r)}$$

$$= \frac{2\sqrt{r}}{c}\left\{\rho\sqrt{(\rho - r)} - \frac{2}{3}(\rho - r)^{\frac{3}{2}}\right\}$$

$$= \frac{2\sqrt{r}}{3c}\sqrt{(\rho - r)} \times (\rho + 2r)$$

But

$$c = PV = \sqrt{2g\mu r} \ (229)$$

$$\therefore t = \frac{\sqrt{2}}{3\sqrt{g\mu}} \times (\rho + 2r)(\rho - r)^{\frac{1}{2}} \quad . \quad . \quad . \quad (b)$$

which gives

$$\rho^3 + 3r\rho^2 = 4r^3 + \frac{9}{2}g\mu t,$$

whence we have ρ and the point required.

By the last Article the value of M in Newton's Assumption is easily obtained, and is

$$M = \frac{A}{4r} = \frac{t}{4} \times \sqrt{\frac{g\mu}{2r}}.$$

231. COR. 1. This readily appears upon drawing S Q the semi-latus-rectum and by drawing through its point of bisection a perpendicular to G H.

232. COR. 2. This proportion can easily be obtained as in the note of the Jesuits, by taking the ratio of the increments of G H and of the curve at the vertex; or the absolute value of the velocity of H is directly got thus.

$$v = \frac{d.GH}{dt} = \frac{3dM}{dt} = \frac{3}{4}\sqrt{\frac{g\mu}{2r}}.$$

Also the velocity in the curve is given by (see 140)

$$v'^2 = 2gF \times \frac{PV}{4} = \frac{2g\mu}{\rho}$$

and at the vertex $\varrho = r$,

$$\therefore v' = \sqrt{\frac{2g\mu}{r}}$$

$$\therefore v : v' :: 3 : 8.$$

233. Cor. 3. Either A P, or S P being bisected, &c. will determine the point H and therefore

$$t = \frac{4}{3}\sqrt{\frac{2r}{g\mu}} \times GH.$$

234. Lemma XXVIII. That an oval cannot be squared is differently demonstrated by several authors. See Vince's Fluxions, p. 356; also Waring.

235. Prop. XXXI. This is rendered somewhat easier by the following arrangement of the proportions:

If G is taken so that

$$OG : OA :: OA : OS$$

or

$$OG = \frac{OA^2}{OS}$$

and

$$GK : 2\pi OG :: t : T$$

or

$$GK = \frac{2\pi \times OA^2}{OS} \times \frac{t}{T} \quad \ldots\ldots\ldots \quad (a)$$

Then, &c. &c. For

$$ASP = ASQ \times \frac{b}{a}$$

$$= \frac{b}{a} \times (OQA - OQS)$$

$$= \frac{b}{2a}(OQ \times AQ - OQ \times SR)$$

$$= \frac{b}{2}(AQ - SR).$$

But

$$SR : \sin. AQ :: SO : OA$$

$$:: OA : OG :: AQ : FG$$

$$\therefore SR = \frac{AQ \sin. AQ}{FG}$$

and

$$AQ - SR = \frac{AQ}{FG} \times (FG - \sin. AQ)$$

$$= \frac{OS}{OA} \times (FG - \sin. AQ)$$

$$\therefore ASP = \frac{OS \times b}{2a} \times (FG - \sin. AQ)$$

$$= \frac{OS \times b}{2a} \times GK \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

(see the Jesuits' note q.) which is identical with (a), since

$$\frac{t}{T} = \frac{ASP}{\text{Ellipse}}$$

$$= \frac{ASP}{\pi a b}.$$

OTHERWISE.

236. By 230 we have

$$dt = \frac{p \varrho d \varrho}{c \sqrt{(\varrho^2 - p^2)}} = \frac{\varrho d \varrho}{c \sqrt{\left(\frac{\varrho^2}{p^2} - 1\right)}}$$

But in the ellipse

$$p^2 = \frac{b^2 \varrho}{2a - \varrho}$$

$$\therefore dt = \frac{b \varrho d \varrho}{c \sqrt{(2a\varrho - b^2 - \varrho^2)}}$$

and putting

$$\varrho - a = u$$

it becomes

$$dt = \frac{b . (a + u) du}{c \sqrt{(a^2 e^2 - u^2)}}$$

2 a e being the excentricity.

Hence

$$t = \frac{ba}{c} \int \frac{du}{\sqrt{(a^2 e^2 - u^2)}} + \frac{b}{c} \int \frac{u du}{\sqrt{(a^2 e^2 - u^2)}}$$

$$= \frac{ba}{c} \sin.^{-1} . \frac{u}{ae} - \frac{b}{c} \sqrt{(a^2 e^2 - u^2)} + C.$$

Let $t = 0$, when $u = ae$; then

$$C = \frac{ba}{c} \times \frac{\pi}{2}$$

and we get

$$t = \frac{ba}{c} \times \left(\frac{\pi}{2} + \sin.^{-1} \frac{u}{ae}\right) - \frac{b}{c} . \sqrt{(a^2 e^2 - u^2)}$$

which is the known form of the equation to the Trochoid, t being the abscissa, &c.

Hence by approximation or by construction u and therefore ϱ may be found, which will give the place of the body in the trajectory.

It need hardly be observed that (157)

$$c = P \times V = \sqrt{\frac{g\mu}{2}}\,L = b\sqrt{\frac{g\mu}{2}}.$$

OTHERWISE.

237. $$dt = \frac{\varrho^2 d\theta}{c};$$

but in the ellipse

$$\varrho = \frac{b^2}{a} \times \frac{1}{1 + e\cos.\theta}$$

$$\therefore dt = \frac{b^4}{a^2 c} \times \frac{d\theta}{(1 + e\cos.\theta)^2}$$

and (see Hirsch's Tables, or art. 110)

$$t = \frac{a^2(1-e^2)}{c} \times \left\{\frac{1}{\sqrt{(1-e^2)}}\cos.^{-1}.\frac{e + \cos.\theta}{1 + e\cos.\theta} - \frac{e\sin.\theta}{1 + e\cos.\theta}\right\}$$

which also indicates the Trochoid.

To simplify this expression let

$$\cos.^{-1}\frac{e + \cos.\theta}{1 + e\cos.\theta} = u$$

then

$$\frac{e + \cos.\theta}{1 + e\cos.\theta} = \cos.u$$

and

$$\cos.\theta = \frac{e - \cos.u}{e\cos.u - 1}$$

Hence

$$\sin.\theta = \frac{\sqrt{(1 - e^2)}}{1 - e\cos.u}$$

and

$$\frac{e\sin.\theta}{1 + e\cos.\theta} = \frac{e\sin.u}{\sqrt{(1-e^2)}}$$

$$\therefore t = \frac{a^2\sqrt{(1-e^2)}}{c} \times \{u - e\sin.u\}$$

But (157)

$$c = PV = b.\sqrt{\frac{g\mu}{a}} = \sqrt{(1-e^2)}\sqrt{g\mu a}$$

$$\therefore t = \frac{a^{\frac{3}{2}}}{\sqrt{g\mu}} \times (u - e \sin. u)$$

Let $$\frac{a^{\frac{3}{2}}}{\sqrt{g\mu}} = \frac{1}{n}.$$

Then

$$n t = u - e \sin. u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, θ may be better expressed in terms of u, thus

$$\tan.^2 \frac{\theta}{2} = \frac{1 - \cos. \theta}{1 + \cos. \theta} = \frac{1 + e}{1 - e} \times \frac{1 - \cos. u}{1 + \cos. u} = \frac{1 + e}{1 - e} \tan.^2 \frac{u}{2}$$

$$\therefore \tan. \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \times \tan. \frac{u}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Moreover ϱ is expressible in terms of u, for

$$\varrho = \frac{a(1 - e^2)}{1 + e \cos. \theta} = a(1 - e \cos. u) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In these three equations, n t is called the *Mean Anomaly*; u the *Excentric Anomaly*, (because it = the angle at the center of the ellipse subtended by the ordinate of the circle described upon the axis-major corresponding to that of the ellipse); and θ the *True Anomaly*.

238. SCHOLIUM.

Newton says that "the approximation is founded on the Theorem that *The area* A P S $\propto$ A Q — S F, S F *being the perpendicular let fall from* S *upon* O Q."]

First we have

$$A P R = A Q R \times \frac{b}{a}$$

$$S P R = S Q R \times \frac{b}{a}$$

$$\therefore A S P = A S Q \times \frac{b}{a}$$

But

$$\begin{aligned} A S Q &= A O Q - S O Q \\ &= \tfrac{1}{2} A Q \times A O - \tfrac{1}{2} S F \times O Q \\ &= \tfrac{1}{2} A O \times (A Q - S F). \end{aligned}$$

$$\therefore A S P = \frac{b}{2} \times (A Q - S F)$$

$$= \frac{b}{2} \times (a u' - a e \sin. u') \quad . \quad . \quad . \quad . \quad . \quad (1)$$

u′ being the $\angle$ A O Q.

(Hence is suggested this easy determination of eq. 1. 237.

For

$$t = T \times \frac{A\,S\,p}{\text{Ellipse}} = \frac{2\,\pi\,a^{\frac{3}{2}}}{\sqrt{\mu\,g}} \times \frac{\frac{b}{2}(a\,u - a\,e\,\sin.\,u)}{\pi\,a\,b}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{g\,\mu}} \times (u - e\,\sin.\,u).)$$

Again, supposing u′ an approximate value of u, let

$$u = u' + \frac{E}{a}$$

Then, by the Theorem, we have

$$\frac{2\,A\,S\,p}{b} = A\,q - S\,O \times \sin.\,A\,q$$

$$= A\,Q + Q\,q + - S\,O \times \sin.\,(A\,Q \pm Q\,q)$$

to radius 1.

But A Q being an approximate value of A q, Q q is small compared with A O, and we have

$$\sin.\,(A\,Q \pm Q\,q) = \sin.\,A\,Q\cos.\,Q\,q \pm \cos.\,A\,Q\sin.\,Q\,q$$

$$= \sin.\,A\,Q \pm Q\,q\cos.\,A\,Q \text{ nearly.}$$

$$\therefore Q\,q = \left(\frac{2\,A\,S\,p}{b} - A\,Q + S\,O\sin.\,A\,Q\right) \times \frac{\frac{1}{SO}}{\frac{1}{SO} \mp \cos.\,A\,Q} \text{ nearly}$$

which points out the use of these assumptions

$$N' = \frac{2\,A\,S\,p}{b} = \frac{2\,t}{b\,T} \times \text{area of the Ellipse}$$

$$B' = S\,O \qquad = \frac{2\,\pi\,a\,t}{T}$$

and

$$D' = S\,O\,.\,\sin.\,A\,Q = B'\sin.\,A\,Q$$

$$L' = \frac{1}{SO}$$

Then

$$Q\,q = (N' - A\,Q + D') \times \frac{L'}{L' \mp \cos.\,A\,Q}$$

in which it is easily seen B′, N′, D′, L′
are identical with B, N, D, L.

Hence

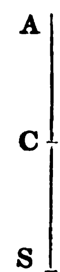
$$E = Q\,q = (N - A\,Q + D)\,\frac{L}{L \mp \cos.\,A\,Q}.$$

Having augmented or diminished the assumed arc A Q by E, then repeat the process, and thus find successively

G, I, &c.

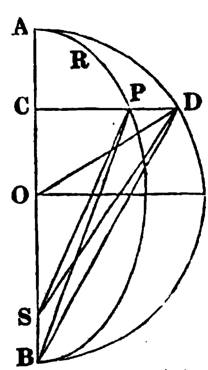
For a developement of the other mode of approximation in this Scholium, see the Jesuits' note 386. Also see Woodhouse's Plane Astronomy for other methods.

SECTION VII.



239. PROP. XXXII. $F \propto \frac{1}{\text{distance}^2}$. Determine the spaces which a body descending from A in a *straight line* towards the center of force describes in a given time.

If the body did not fall in a straight line to the center, it would describe some conic section round the center of force, as focus (which would be $\left\{\begin{array}{c}\text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{array}\right\}$ if the velocity at any point were to the velocity in the circle, the same distance and force, in $R^o.\left\{\begin{array}{c}<\\ =\\ >\end{array}\right\}$ $\sqrt{2} : 1$.)



(I) Let the Conic Section be an Ellipse A R P B.

Describe a circle on Major Axis A B, draw C P D through the place of the body perpendicular to A B.

The time of describing A P $\propto$ area A S P $\propto$ area A S D, whatever may be the excentricity of the ellipse.

Let the Axis Minor of the ellipse be diminished sine limite and the ellipse becomes a straight line ultimately, A B being constant, and since A S . S B = (Minor Axis)^2 = 0, and A S finite $\therefore$ S B = 0, or B ultimately comes to S, and time d . A C $\propto$ area A D B. $\therefore$ if A D B be taken proportional to time, C is found by the ordinate D C.

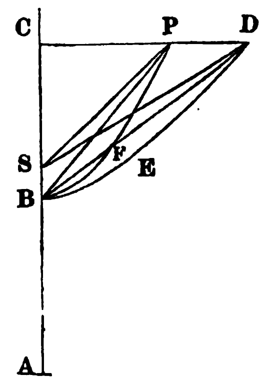
(T . A C $\propto$ area A D B $\propto$ A D O + O D B $\propto$ arc A D + C D

$\therefore$ take θ + sin. θ proportional to time, and D and C are determined.)

(Hence

$$\frac{\text{the time down AO}}{\text{T.OB}} = \frac{\frac{\pi}{2}+1}{\pi-\frac{\pi}{2}+1} = \frac{\frac{\pi}{2}+1}{\frac{\pi}{2}-1} = \frac{\frac{11}{7}+1}{\frac{11}{7}-1} = \frac{18}{4} = \frac{9}{2} \text{ nearly})$$

N. B. The time in this case is the time from the beginning of the fall, or the time from A.



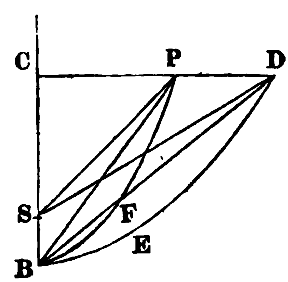
(II) Let the conic section be the hyperbola B F P. Describe a rectangular hyperbola on Major Axis A B.

T $\propto$ area S B F P $\propto$ area S B E D.

Let the Minor Axis be diminished sine limite, and the hyperbola becomes a straight line, and T $\propto$ area B D E.

N. The time in this case is the time from the end of motion or time to S.

Let the conic section be the parabola B F P. Describe any fixed parabola B E D.



T $\propto$ area S B F P $\propto$ area S B E D.

Let L. R. of B F P be diminished sine limite the parabola becomes a straight line, and T $\propto$ area B D E.

N. The time in this case is the time from the end of motion, or time to S.

Objection to Newton's method. If a straight line be considered as an evanescent conic section, when the body comes to perihelion i. e. to the center it ought to return to aphelion i. e. to the original point, whereas it will go through the center to the distance below the center = the original point.

240. We shall find by Prop. XXXIX, that the distance from a center from which the body must fall, acted on by $\propto^{\text{ble}}$ force, to acquire the velocity such as to make it describe an ellipse = A B (finite distance), for the hyperbola = — A B, for the parabola = α.

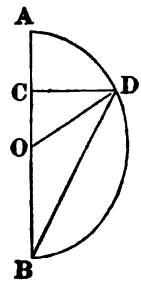
241. Case 1. $v\,dv = -g\mu \frac{dx}{x^2}$, f = force distance 1,

$$\therefore v^2 = g\mu.\left(\frac{a-x}{ax}\right)$$

if a be the original point

$$dt = -\frac{dx}{v} = -\frac{\sqrt{a}}{\sqrt{2g\mu}}dx.\frac{x}{\sqrt{ax-x^2}}$$

$$= \sqrt{\frac{a}{2g\mu}} \cdot \left\{ \frac{\left(\frac{a}{2} - x\right) dx}{\sqrt{ax - x^2}} \quad \frac{-\frac{a}{2} dx}{\sqrt{ax - x^2}} \right\}$$



$$\therefore t = \sqrt{\frac{a}{2g\mu}} \cdot \left\{ \sqrt{ax - x^2} - \text{vers.}^{-1} x \atop \text{rad.} = \frac{a}{2} \right\}$$

$$+ C, \text{ when } t = 0, x = a,$$

$$\therefore t = \sqrt{\frac{a}{2g\mu}} \cdot \left\{ \sqrt{ax - x^2} + \left(\frac{\text{circumference}}{2} - \text{vers.}^{-1} x, \atop \text{rad.} = \frac{a}{2} \right) \right\}$$

$$= \sqrt{\frac{a}{2g\mu}} (CD + AD)$$

if the circle be described on $BA = a$,

$$= \sqrt{\frac{a}{2g\mu}} \cdot \frac{4}{a} \cdot \left(\frac{CD.OB}{2} + \frac{AD.OD}{2} \right) = \frac{2}{\sqrt{2g\mu}} . BAD.$$

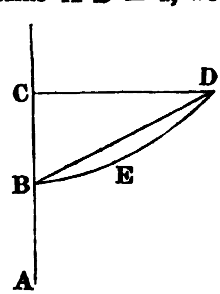
Case 2. $v^2 = 2g\mu . \frac{a+x}{ax}$; if $-a$ be an original point,

$$\therefore t = \sqrt{\frac{a}{2g\mu}} . \int \frac{x\,dx}{\sqrt{ax + x^2}},$$

for t in this case is the time to the center, not the time from the original point,

$$\therefore -dt = -\frac{dx}{v}, \text{ or } dt = \frac{dx}{v}.$$

Now if with the Major Axis $AB = a$, we describe the rectangular hyperbola,

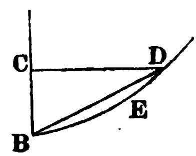


we have

$$d.BED = d.BEDC - d.ABDC = y\,dx - \frac{d.xy}{2} = \frac{y\,dx - x\,dy}{2}$$

$$= \frac{\sqrt{ax+x^2}}{2}.dx - \frac{x.\left(\frac{a}{2}+x\right)dx}{2\sqrt{ax+x^2}} = \frac{a\,x\,dx}{4\sqrt{ax+x^2}} = dt.\frac{\sqrt{ag\mu}}{2\sqrt{2}}$$

$\therefore$ t from B $= \sqrt{\frac{2}{2g\mu}}$. B E D, for they begin or end together at B.



Case 3. $v^2 = 2g\mu\frac{1}{x}$, if a be α,

$$\therefore dt = \frac{dx}{v} = \frac{\sqrt{x}.dx}{\sqrt{2g\mu}}, \text{ t being time to B,}$$

$$\therefore t = \frac{1}{\sqrt{2g\mu}}\frac{2}{3}.x^{\frac{3}{2}} + C, \text{ when } t = 0, x = 0, \therefore C = 0.$$

Describe a parabola on the line of fall, vertex B, L. R. = any fixed distance a,

$$\therefore t = \frac{1}{\sqrt{2g\mu}}.\tfrac{1}{3}.\sqrt{x}.x = \frac{2\sqrt{2}}{\sqrt{ag\mu}}.\tfrac{1}{6}.\sqrt{ax}.x = \frac{2\sqrt{2}}{\sqrt{ag\mu}}.BED.$$

Hence in general, in Newton Prop. XXXII, $t = \frac{2\sqrt{2}}{\sqrt{ag\mu}}$. curvilinear area, a being L. R. of the figure described.

In the evanescent conic sections, L. R. $= \frac{2\,(\text{Ax. Min.})^2}{\text{Ax. Maj.}}$, $\therefore$ if Ax. Min. be indefinitely small, L. R. will be indefinitely small with respect to the Ax. Min. The chord of curvature at the finite distance from A to B is ultimately finite, for $PV = \frac{2\,SP.AP}{\frac{AB}{2}}$; but at A or B, PV = L, = infinitesimal of the second order. Hence S B is also ultimately of the second order, for at B, $SB = L.\frac{\frac{AB}{2}}{2AS}$.

Prop. XXXIII. Force $\alpha\ \frac{1}{(\text{distance})^2}$.

$$\frac{\text{V at C}}{\text{v in the circle distance SC}} = \frac{\sqrt{AC}}{\sqrt{\frac{SA}{2}}} \text{ in the ellipse and hyperbola.}$$

$\left(\frac{V}{v}=\frac{\sqrt{HP}}{\sqrt{\frac{\text{Maj. Ax.}}{2}}}=\frac{\sqrt{AC}}{\sqrt{\frac{SA}{2}}}\text{ when the conic section becomes a straight line}\right)$

NEWTON'S METHOD.

$$\frac{V^2}{v^2}=\frac{\frac{L}{SY^2}}{\frac{2SP}{SP^2}}=\frac{L}{2}\cdot\frac{SP}{SY^2}$$

$$\frac{AC.CB}{CP^2}=\frac{AO^2}{\left(\frac{\text{Min. Ax.}}{2}\right)^2}=\frac{2AO}{\frac{2\left(\frac{\text{Min. Ax.}}{2}\right)^2}{AO}}=\frac{2AO}{L}$$

$$\therefore\frac{L}{2}=\frac{AO.CP^2}{AC.CB}$$

$$\therefore\frac{V^2}{v^2}=\frac{AO.CP^2.SP}{AC.CB.SY^2},$$

but

$$\frac{CO}{BO}=\frac{BO}{TO},$$

$$\therefore\frac{CO}{BO}=\frac{CB}{BT},\ \text{comp. in the ellipse}\atop\text{div. in the hyperbola},$$

$$\therefore\frac{AC}{BO}=\frac{CT}{BT},\ \text{div. in the ellipse}\atop\text{comp. in the hyperbola}\ =\frac{CP}{BQ},$$

$$\therefore\frac{AC^2}{AO^2}=\frac{CP^2}{BQ^2},$$

$$\therefore\frac{BQ^2.AC}{AO}=\frac{AO.CP^2}{AC},$$

$$\therefore\frac{V^2}{v^2}=\frac{BQ^2.AC.SP}{AO.BC.SY^2},$$

but ultimately

$$BQ=SY,\ SP=BC,$$

$$\therefore\text{ ultimately }\frac{V^2\text{ in a straight line}}{v^2\text{ in the circle}}=\frac{AC}{AO},$$

$$\therefore\frac{V}{v}=\sqrt{\frac{AC}{AO}}.$$

COR. 1. It appeared in the proof that $\frac{AC}{AO}=\frac{CT}{BT}$,

$$\therefore \text{ ultimately } \frac{AC}{AO} = \frac{CT}{ST}.$$

(This will be used to prove next Prop.)

Cor. 2. Let C come to O, then A C = A O and V = v,

$\therefore$ the velocity in the circle = the velocity acquired by falling externally through distance = rad. towards the center of the force $\propto \frac{1}{\text{distance}^2}$.

242. *Find actual Velocity at* C.

$$\frac{V^2 \text{ at C}}{v^2 \text{ in the circle distance BC}} = \frac{AC}{\frac{BA}{2}},$$

$$\therefore V^2 = \frac{2\,AC}{BA} \cdot v^2 = \frac{2\,AC}{BA} \cdot \frac{g\mu}{BC^2} \cdot BC,$$

if μ = the force at distance 1,

$$\therefore V^2 = 2\,g\,\mu \frac{AC}{BA \cdot BC},$$

$$\therefore V = \sqrt{2\,g\,\mu} \cdot \frac{\sqrt{a-x}}{\sqrt{a\,x}}, \text{ if BA} = a, \text{ BC} = x.$$

If a is given, $V \propto \frac{\sqrt{\text{space described}}}{\sqrt{\text{space to be described}}}$.

In descents from different points,

$$V \propto \frac{\sqrt{\text{space described}}}{\sqrt{\text{space to be described} \times \text{initial height}}}.$$

In descents from different points to different centers,

$$V \propto \frac{\sqrt{\text{space described} \times \text{absolute force}}}{\sqrt{\text{space to be described} \times \text{initial height}}}$$

243. Otherwise. $v\,dv = -\frac{g\mu}{x^2}\,dx$,

$$\therefore v^2 = 2\,g\,\mu \cdot \frac{a-x}{a\,x}, \text{ when a is positive, as in the ellipse}$$

$$= 2\,g\,\mu \cdot \frac{a+x}{a\,x}, \text{ when a is negative as in the hyperbola}$$

$$= 2\,g\,\mu \cdot \frac{1}{x}, \text{ when a is } \propto, \text{ as in parabola}$$

(when $x = 0$, v is infinite)

V^2 in the circle radius x (in the ellipse and hyperbola)

$$= \frac{g\mu}{x^2} \cdot x = \frac{g\mu}{x}$$

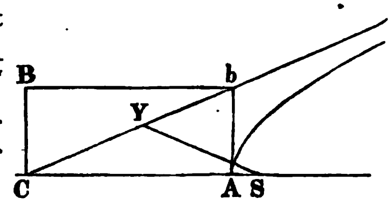
$$\therefore \frac{v^2}{V^2} = \frac{2\,\overline{a-x}}{a} \text{ in the ellipse}, = \frac{a-x}{\left(\frac{a}{2}\right)}.$$

$$\frac{v^2}{V^2} = \frac{2\,\overline{a+x}}{a} \text{ in the hyperbola, } = \frac{a+x}{\left(\frac{a}{2}\right)}$$

$$V^2 \text{ in the circle radius } = \frac{x}{2} \text{ (in the parabola) } = \frac{g\mu}{\frac{x^2}{4}} \cdot \frac{x}{2} = \frac{2g\mu}{x}$$

$$\therefore \frac{v^2}{V^2} = \frac{1}{1} \text{ in the parabola.}$$

244. In the hyperbola not evanescent



$\frac{\text{Velocity at the infinite distance}}{\text{velocity at A}} = \frac{S\,A}{S\,Y}$ finite R°., but when the hyperbola vanishes, S Y ultimately = Min. Ax. for $\frac{S\,Y}{A\,b} = \frac{S\,C}{b\,C}$, and ultimately S C = A C, and b C = A C, ∴ ultimately S Y = A b = C B, ∴ ultimately

$$\frac{S\,Y}{S\,A} = \frac{\text{infinitesimal of the first order}}{\text{——— of the 2d order}}$$

$$= \frac{\text{velocity at A}}{\text{velocity at } \infty \text{ distance}}$$

245. PROP. XXXIV. $\frac{\text{Velocity at C}}{\text{velocity in the circle, distance } \frac{S\,C}{2}} = \frac{1}{1}$, for the parabola.

For the velocity in the parabola at P = velocity in the circle $\frac{S\,P}{2}$ whatever be L. R. of the parabola.

246. PROP. XXXV. Force $\propto \frac{1}{(\text{distance})^2}$.

The same things being assumed, the area swept out by the indefinite radius S D in fig. D E S = area of a circular sector (rad. = $\frac{L.R}{2}$ of fig.) uniformly described about the center S in the same time. Whilst the falling body describes C c indefinitely small, let K k be the arc described by the body uniformly revolving in the circle.

Case 1. If D E S be an ellipse or rectangular hyperbola, $\frac{L.R}{2} = \frac{S\,A}{2}$,

$$\frac{C\,c}{D\,d} = \frac{C\,T}{D\,T}$$

$$\frac{C\,D}{S\,Y} = \frac{D\,T}{T\,S}$$

$$\therefore \frac{Cc.CD}{Dd.SY} = \frac{CT}{TS} = \frac{AC}{AO} \text{ ultimately.}$$

(Cor. Prop. XXXIII.)

But

$$\frac{\text{velocity at C}}{\text{v in the circle rad. SC}} = \frac{\sqrt{AC}}{\sqrt{AO}}$$

$$\frac{\text{v in the circle rad. SC}}{\text{v in the circle rad. SK}} = \sqrt{\frac{SK}{SC}} = \sqrt{\frac{AO}{SC}}$$

$$\therefore \left(\frac{\text{velocity at C}}{\text{v in the circle rad. SK}}\right) = \frac{Cc}{Kk} = \sqrt{\frac{AC}{SC}} = \frac{AC}{CD}$$

$$\therefore Cc.CD = Kk.AC$$

$$\therefore \frac{Kk.AC}{Dd.SY} = \frac{AC}{AO},$$

$$\therefore AO.Kk = Dd.SY,$$

$\therefore$ the area S K k = the area S D d,

$\therefore$ the nascent areas traced out by S D and S K are equal

$\therefore$ the sums of these areas are equal.

Case 2. If D E S be a parabola S K $= \frac{L.R}{2}$.

As above

$$\frac{Cc.CD}{Dd.SY} = \frac{CT}{TS} = \frac{2}{1}$$

also

$$\frac{Cc}{Kk} = \frac{\text{velocity at C}}{\text{velocity in the circle } \frac{L.R}{2}} = \frac{\text{velocity in the circle } \frac{SC}{2}}{\text{velocity in the circle } \frac{L.R}{2}}$$

$$= \frac{\sqrt{SK}}{\sqrt{\frac{SC}{2}}} = \frac{SK}{\frac{CD}{2}}$$

$$\therefore Cc.CD = 2.Kk.SK$$

$$\therefore Kk.SK = Dd.SY.$$

247. Prop. XXXVI. Force $\propto \frac{1}{(\text{distance})^2}$.

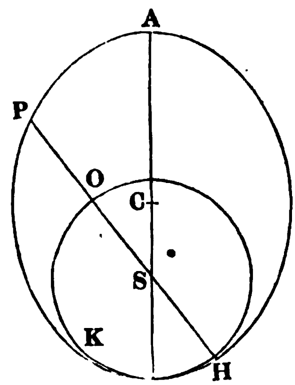
To determine the times of descent of a body falling from the given (and $\therefore$ finite) altitude A S

On A S describe a circle and an equal circle round the center S.

From any point of descent C erect the ordinate C D, join S D. Make the sector O S K = the area A D S (O K = A D + D C) the body will fall from A to C in the time of describing O K about the center S

uniformly, the force $\propto \frac{1}{\text{distance}^2}$. Also S K being given, the period in the circle may be found, $\left(P = \sqrt{\frac{4\mu}{g}} \cdot \pi \cdot SK^{\frac{3}{2}}\right)$, and the time through $OK = P \cdot \frac{OK}{\text{circumference}}$. ∴ the time through O K is known. ∴ the time through A C is known.

248. *Find the time in which a Planet would fall from any point in its orbit to the Sun.*



Time of fall = time of describing $\frac{\text{circle}}{2}$ O K H, $SO = \frac{SP}{2}$,

$$\frac{\text{period in the circle } OKH}{\text{period in the ellipse}} = \frac{\text{period in the circle rad. } SO}{\text{period in the circle rad. } AC} = \frac{SO^{\frac{3}{2}}}{AC^{\frac{3}{2}}}$$

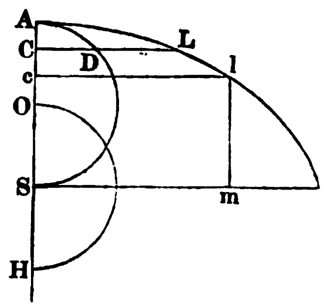
∴ the time of fall $= \frac{1}{2} \cdot P \cdot \left(\frac{SO}{AC}\right)^{\frac{3}{2}}$, P = period of the planet. If the orbit be considered a circle

$$\left(\frac{SO}{AC}\right)^{\frac{3}{2}} = \left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{1}{\sqrt{8}}$$

and the time of fall

$$= \frac{P}{4\sqrt{2}} = P \cdot \frac{\sqrt{2}}{8} = P \cdot \frac{\frac{4}{3}}{8} \text{ nearly.}$$

$$= \frac{P}{6} \text{ nearly.}$$



249. The time down A C $\propto$ (arc $=$ A D $+$ C D), $\propto$ C L, if the cycloid be described on A S. Hence, having given the place of a body at a given time, we can determine the place at another given time.

Cut off $S\,m = C\,L \cdot \frac{\text{time d. } A\,c}{\text{time d. } A\,C}$.

Draw the ordinate m l; l c will determine c the place of the body.

250. Prop. XXXVII. *To determine the times of ascent and descent of a body projected upwards or downwards from a given point*, F $\propto \frac{1}{\text{distance}^2}$.

Let the body move off from the point G with a given velocity. Let $\frac{V^2 \text{ at } G}{v^2 \text{ in the circle e. d.}} = \frac{m^2}{1}$, (V and v known, $\therefore$ m known).

To determine the point A, take

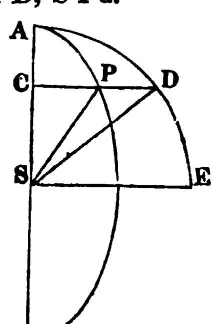
$$\frac{G\,A}{\frac{S\,A}{2}} = \frac{m^2}{1}$$

$$\therefore \frac{G\,A}{G\,A + G\,S} = \frac{m^2}{2}$$

$$\therefore \frac{G\,A}{G\,S} = \frac{m^2}{2 - m^2}$$

$\therefore$ if $m^2 = 2$, G A is $+$ and ∞, $\therefore$ the parabola
if $m^2 < 2$, G A is $+$ and fin. $\therefore$ the circle
if $m^2 > 2$, G A is $-$ and fin. $\therefore$ the rectangular hyperbola
} must be described on the axis S A.

With the center S and rad. $= \frac{L}{2}$ of the conic section, describe the circle k K H, and erecting the ordinates G I, C D, c d, from any places of the body, the body will describe G C, G c, in times of describing the areas S K k, S K k′, which are respectively $=$ S I D, S I d.



251. Prop. XXXVIII. Force $\propto$ distance.

Let a body fall from A to any point C, by a force tending to S, and $\propto^s$. as the distance. Time $\propto$ arc A D, and V acquired $\propto$ C D. Conceive a body to fall in an evanescent ellipse about S as the center.

$\therefore$ the time down A P or A C

$\propto$ A S P $\propto$ A S D $\propto$ A D. $\frac{A\,S}{2}$

$\propto$ A D for the same descent, i. e. when A is given.

The velocity at any point P

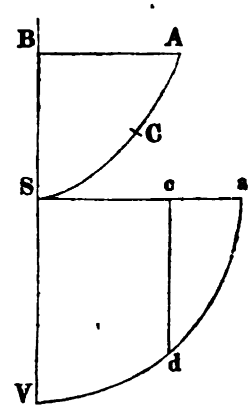
$$\propto \sqrt{F.PV}$$

$$\propto \sqrt{SP.\frac{2AC.Ca}{SP}} \text{ ultimately.}$$

$$\propto CD.$$

COR. 1. T. from A to S = $\frac{1}{4}$ period in an evanescent ellipse.
= $\frac{1}{4}$ period in the circle A D E.
= T. through A E.

COR. 2. T. from different altitudes to S $\propto$ time of describing different quadrants about S as the center $\propto$ 1.



N. In the common cycloid A C S it is proved in Mechanics that if S c a = S C A and the circle be described on 2. S c a, and if a c = A C, the space fallen through, then the time through A C $\propto$ arc a d, and V acquired $\propto$ c d, which is analogous to Newton's Prop.

Newton's Prop. might be proved in the same way that the properties of the cycloid are proved.

OTHERWISE.

252. $v\,dv = -g\mu x.dx,$

$\therefore v^2 = 2g\mu(a^2 - x^2)$, if a = the height fallen from

$$\therefore v = \sqrt{2g\mu}.\sqrt{a^2-x^2} = \sqrt{2g\mu}.CD.$$

$$dt = -\frac{dx}{v} = -\frac{dx}{\sqrt{2g\mu}}.\frac{1}{\sqrt{a^2-x^2}}$$

$$\therefore t = +\frac{\text{arc}}{a\sqrt{2g\mu}}.\binom{\text{cos.} = x}{\text{rad.} = a} + C,\ C = 0,$$

$$= \frac{1}{a\sqrt{2g\mu}}.AD.$$

$\therefore$ velocity $\propto$ *sine* of the arc whose *versed sine* = space, and the *arc* $\propto$ time, (rad. = original distance.)

253. *The velocity is velocity from a finite altitude.*

If the velocity had been that from infinity, it would have been infinite

and constant. $\therefore \mathrm{d\,t} = -\frac{\mathrm{d\,x}}{\mathrm{v}}$, and $\mathrm{t} = \frac{\mathrm{x}}{\mathrm{a}.\sqrt{\mathrm{g}\,\mu}} + \mathrm{C}$, when $\mathrm{t} = 0$,

$$= \sqrt{\mathrm{g}\,\mu}\,.\,\mathrm{a},\ \mathrm{a} = \alpha\,.$$

$\mathrm{x}=\mathrm{a}$, $\therefore$ c is finite, $\therefore \mathrm{t} = \mathrm{C} = \frac{1}{\sqrt{\mathrm{g}\,\mu}}$.

Similarly if the velocity had been $>$ velocity from infinity, it would have been infinite.

254. Prop. XXXIX. *Force* $\propto$ *(distance)*$^{\mathrm{n}}$, *or any function of distance.*

Assuming any $\propto^{\mathrm{n}}$. of the centripetal force, and also that quadratures of all curves can be determined (i. e. that all fluents can be taken); Required the velocity of a body, when ascending or descending perpendicularly, at different points, and the time in which a body will arrive at any point.

(The proof of the Prop. is inverse. Newton assumes the area A B F D to $\propto \mathrm{V}^2$ and A D to $\propto$ space described, whence he shows that the force $\propto$ D F the ordinate. Conversely, he concludes, if F $\propto$ D F, A B F D $\propto \mathrm{V}^2$.)

$$\mathrm{v}^2 \propto \int \mathrm{v\,d\,v} \propto \int \mathrm{F}\,.\,\mathrm{d\,s}.$$

Let D E be a small given increment of space, and I a corresponding increment of velocity. By hypothesis

$$\frac{\mathrm{A\,B\,F\,D}}{\mathrm{A\,B\,G\,E}} = \frac{\mathrm{V}^2}{\mathrm{v}'^2} = \frac{\mathrm{V}^2}{\mathrm{V}^2 + 2\,\mathrm{V}.\mathrm{I} + \mathrm{I}^2}$$

$$\therefore \frac{\mathrm{A\,B\,F\,D}}{\mathrm{D\,F\,G\,E}} = \frac{\mathrm{V}^2}{2\,\mathrm{V}.\mathrm{I} + \mathrm{I}^2} = \frac{\mathrm{V}^2}{2\,\mathrm{V}.\mathrm{I}} \text{ ultimately.}$$

But

$$\mathrm{A\,B\,F\,D} \propto \mathrm{V}^2 \ \therefore \mathrm{D\,F\,G\,E} \propto 2\,\mathrm{V}.\mathrm{I}$$

$$\therefore \mathrm{D\,E}.\mathrm{D\,F} \text{ ultimately. } \propto 2.\mathrm{V}.\mathrm{I}$$

$$\therefore \mathrm{D\,F} \propto \frac{2\,\mathrm{V}.\mathrm{I}}{\mathrm{D\,E}} \propto \frac{\mathrm{I}.\mathrm{V}}{\mathrm{D\,E}}.$$

But in motions where the forces are constant if I be the velocity generated in T, $\mathrm{F} \propto \frac{\mathrm{I}}{\mathrm{T}}$, $\left(\mathrm{F} \propto \frac{\mathrm{d\,v}}{\mathrm{d\,t}}\right)$ and if S be the space described with uniform velocity V in T, $\frac{\mathrm{V}}{\mathrm{S}} = \frac{1}{\mathrm{T}}$, $\left(\mathrm{d\,t} = \frac{\mathrm{d\,s}}{\mathrm{v}}\right)$. Also when the force is $\propto^{\mathrm{ble}}$, the same holds for nascent spaces. $\therefore \mathrm{F} \propto \frac{\mathrm{I}.\mathrm{V}}{\mathrm{S}}$, and D E represents S. $\therefore$ D F represents F.

Let $DL \propto \frac{1}{\sqrt{ABFD}} \propto \frac{1}{V}$, $\therefore$ D L M E ultimately $= DL.DE$

$\propto \frac{DE}{V} \propto$ time through D E ultimately.

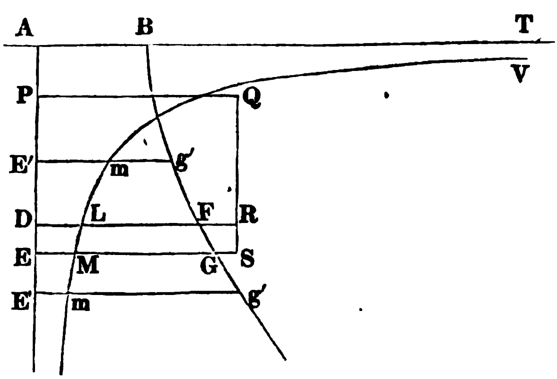
$\therefore$ Increment of the area A T V M E $\propto$ increment of the time down A D.

$\therefore$ A T V M E $\propto$ T.

$$\int dt \propto \int \frac{ds}{v}$$

(Since A B F D vanishes at A, $\therefore$ A T is an asymptote to the time curve. And since E M becomes indefinitely small when A B F D is infinite, $\therefore$ A E is also an asymptote.)

255. Cor. 1. Let a body fall from P, and be acted on by a constant force given. If the velocity at D = the velocity of a body falling by the action of $\propto^{ble}$ force, then A, the point of fall, will be found by making A B F D = P Q R D.



For

$$\frac{ABFD}{DFGE} = \frac{V}{2I} \text{ by Prop.}$$

$$\frac{DFGE}{DRSE} = \frac{DF}{DR} = \frac{I}{i}$$

if i be the increment of the velocity generated through D E by a constant force.

$$\frac{DRSE}{PQRD} = \frac{V^2(V+i)^2}{V^2} = \frac{2i}{V} \text{ ultimately.}$$

$$\therefore \frac{ABFD}{PQRD} = \frac{1}{1}.$$

256. Cor. 2. If a body be projected up or down in a straight line from the center of force with a given velocity, and the law of force given; Find the velocity at any other point E′. Take E′ g′ for the force at E′.

velocity at E′ = velocity at D. $\frac{\sqrt{PQRD \pm DFg'E'}}{\sqrt{PQRD}}$, + if projected down, — if projected up.

$$\left(\text{For } \frac{\sqrt{PQRD \pm DFg'E'}}{\sqrt{PQRD}} = \frac{\sqrt{ABg'E'}}{\sqrt{ABFD}}\right).$$

257. Cor. 3. Find the time through D E′.

Take E′ m inversely proportional to $\sqrt{PQRD \pm DFg'E'}$ (or to the velocity at E′).

$$\frac{T.PD}{T.PE} = \frac{\sqrt{PD}}{\sqrt{PE}} = \frac{\sqrt{PD}}{\sqrt{(PD+DE)}} = \frac{\sqrt{PD}}{\sqrt{PD} + \frac{DE}{2\sqrt{PD}}} \text{ (D E small)}$$

$$= \frac{PD}{PD + \frac{DE}{2}}$$

$$\therefore \frac{T.PD}{T.DE} = \frac{2PD}{DE} = \frac{2PD.DL}{DLME}$$

also

$$\frac{T.DE \text{ by } \alpha^{ble} \text{ force}}{T.DE' \text{ by do.}} = \frac{DLME}{DLmE'},$$

but T . D E by a constant force = T . D E by α^{ble} force since the velocities at D are equal $\left(dt = \frac{ds}{v}\right)$

$$\therefore \frac{T.PD}{T.DE'} = \frac{2PD.DL}{DLmE'}.$$

258. It is taken for granted in Prop. XXXIX, that F α $\frac{dv}{dt}$ (46), and that $v = \frac{ds}{dt}$, whence it follows that if $c . F = \frac{dv}{dt}$, $dv = c . F . dt$, and $v\,dv = c\,F . ds$.

$$\therefore v^2 = 2c\int F\,ds$$

Newton represents $\int F\,ds$ by the area A B F D, whose ordinate D F always = F.

$$dt = \frac{ds}{v} = \frac{ds}{\sqrt{2c.\int F\,ds}},$$

$$\therefore t = \int \frac{ds}{\sqrt{2c\int F\,ds}}.$$

Newton represents $\int \frac{ds}{\sqrt{\int F\,ds}}$ by the area A B T U M E, whose ordinate D L always $= \frac{1}{V}$

$$\left(= \frac{1}{\sqrt{2g \,.\, ABFD}}\right).$$

In COR. 1. If F′ be a constant force $V^2 = 2g\,F'\,.\,PD$, by Mechanics but

$$V^2 = 2c \,.\int F\,ds$$

And $F'\,.\,PD$ or $PQRD$ is proved $= \int F\,ds$ or A B F D,

$$\therefore c = g$$

and

$$v^2 = 2g\,.\int F\,ds.$$

In COR. 2. $\frac{\text{velocity at } E'}{\text{velocity at } D} = \frac{\sqrt{\int F\,ds} \text{ when } s = AE'}{\sqrt{\int F\,ds} \text{ when } s = AD}$

$$= \frac{\sqrt{ABg'E'}}{\sqrt{ABFD}}.$$

In COR. 3. $t = \text{time through } DE' = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2g\int F\,ds}} = DLmE'$,

$$T = \text{time through } PD = \frac{2PD}{V \text{ at } D} = \frac{2PD}{\sqrt{2g\,.\,PQRD}}$$
$$= 2PD\,.\,DL$$

$$\therefore \frac{T}{t} = \frac{2PD\,.\,DL}{DLmE'}.$$

259. The force $\propto x^n$.

$\therefore v\,dv = -g\,\mu\,x^n\,dx$, μ = the force distance 1.

$$\therefore v^2 = \frac{2g\mu}{n+1}\left(a^{n+1} - x^{n+1}\right)$$

if a be the original height.

Let n be positive.

V from a finite distance to the center is finite
V from ∞ to a finite distance is infinite.

Let n be negative but less than 1.

V from a finite distance to the center is finite
V from ∞ to a finite distance is infinite.

Let $n = -1$ the above Integral fails, because x disappears, which cannot be.

$$v\,d\,v = -g\,\mu\,\frac{d\,x}{x}$$

$$\therefore v^2 = 2\,g\,\mu\,.\,l\,\frac{a}{x}$$

$\therefore$ V from a finite distance to the center is infinite
V from ∞ to a finite distance is infinite.

But the log. of an infinite quantity is ∞^{ly} less than the quantity itself $\frac{l\,x}{x}$ when x is infinite $= \frac{0}{0}$. Diff. and it becomes $\frac{\frac{d\,x}{x}}{d\,x} = \frac{1}{x} = \frac{1}{\infty}$.

Let n be negative and greater than 1.

V from a finite distance to the center is infinite
V from ∞ to a finite distance is finite.

260. If the force be constant, the velocity-curve is a straight line parallel to the line of fall, as Q R in Prop. XXXIX.

DEDUCTIONS.

261. To find under what laws of force the velocity from ∞ to a finite distance will be infinite or finite, and from a finite distance to the center will be finite or infinite.

If (1) $F \propto x^2$, $V \propto \sqrt{a^3 - x^3}$
(2) —— x —— $\sqrt{a^2 - x^2}$
(3) —— 1 —— $\sqrt{a - x}$
(4) —— $\frac{1}{x}$ —— $\sqrt{.\,l\,\frac{a}{x}}$
(5) —— $\frac{1}{x^2}$ —— $\sqrt{\frac{a - x}{a\,x}}$
(6) —— $\frac{1}{x^3}$ —— $\sqrt{\frac{a^2 - x^2}{a^2\,x^2}}$
(7) —— $\frac{1}{x^n}$ —— $\sqrt{\frac{a^{n-1} - x^{n-1}}{a^{n-1}\,x^{n-1}}}$

In the former cases, or in all cases where F $\propto$ some direct power of distance, the velocity acquired in falling from ∞ to a finite distance or to the center will be infinite, and from a finite distance to the center will be finite.

In the 4th case, the velocity from ∞ to a finite, and from a finite distance to the center will be infinite.

In the following cases, when the force $\propto$ as some inverse power of distance, the velocity from ∞ to a finite distance will be finite, for

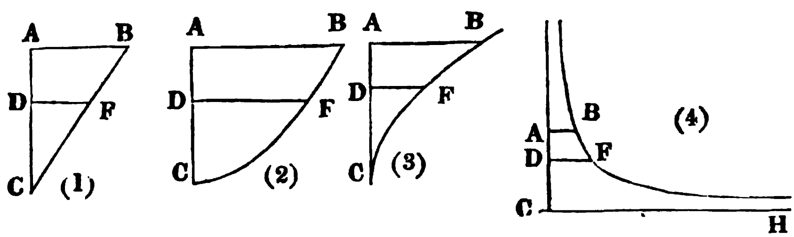
$$\sqrt{\frac{a^{n-1} - x^{n-1}}{a^{n-1} x^{n-1}}} = \sqrt{\frac{1}{x^{n-1}}}$$

when a is infinite. And the velocity from a finite distance to the center will be infinite, for

$$\sqrt{\frac{a^{n-1} - x^{n-1}}{a^{n-1} x^{n-1}}} = \sqrt{\frac{1}{0}}$$

when $x = 0$.

262. *On the Velocity and Time-Curves.*



(1) Let $F \propto D$, the area which represents V^2 becomes a $\triangle$.
For $DF \propto DC$.

(2) Let $F \propto \sqrt{D}$, $\therefore DF^2 \propto DC$ and V-curve is a parabola.

(3) Let $F \propto D^2$, $\therefore DF \propto DC^2$, and V-curve is a parabola the axis parallel to A B.

(4) Let $F \propto \frac{1}{D}$, $\therefore DF \propto \frac{1}{DC}$, $\therefore$ V-curve is an hyperbola referred to the asymptotes A C, C H.

(5) If $F \propto D$, and be repulsive, $V^2 \propto DC \,.\, DF \propto DC^2$, $\therefore V \propto DC$, $\therefore$ the ordinate of the time curve $\propto \frac{1}{V} \propto \frac{1}{DC}$, $\therefore$ T-curve is an hyperbola between asymptotes.

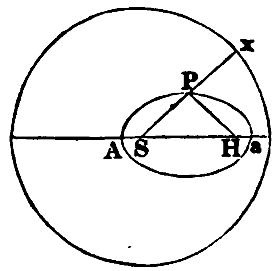
(6) If a body fall from ∞ distance, and $F \propto \frac{1}{D^3}$, $V \propto \frac{1}{D}$, $\therefore$ the ordinate of the time-curve D, $\therefore$ T-curve is a straight line.

(7) If a body fall from ∞, and $F \propto \frac{1}{D^2}$, $V \propto \frac{1}{\sqrt{D}}$, $\therefore$ the ordinate of T-curve $\propto \sqrt{DC}$, $\therefore$ T-curve is a parabola.

(8) If a body fall from ∞, and $F \propto \frac{1}{D^5}$, $V \propto \frac{1}{D^2}$, $\therefore$ the ordinate of T-curve $\propto DC^2$, $\therefore$ T-curve is a parabola as in case 3.

EXTERNAL AND INTERNAL FALLS.

263. *Find the external fall in the ellipse, the force in the focus.*



Let x P be the space required to acquire the velocity in the curve at P.

$$\frac{\text{V}^2 \text{ down P x}}{\text{V}^2 \text{ in the circle distance S P}} = \frac{\text{P x}}{\frac{\text{S x}}{2}}$$

$$\frac{\text{V}^2 \text{ in the circle distance S P}}{\text{V}^2 \text{ in the ellipse at P}} = \frac{\text{A a}}{2 . \text{H P}}$$

$$\therefore \frac{\text{V}^2 \text{ down P x}}{\text{V}^2 \text{ in the ellipse at P}} = \frac{\text{A a . P x}}{\text{S x . H P}}$$

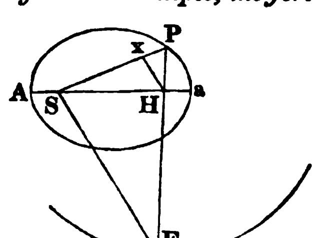
$$\therefore \frac{\text{P x}}{\text{S x}} = \frac{\text{H P}}{\text{A a}}$$

$$\therefore \frac{\text{P x}}{\text{S P}} = \frac{\text{H P}}{\text{S P}}$$

$$\therefore \text{P x} = \text{H P}$$

$\therefore$ S x = S P + P x = A a, and the locus of x is the circle on 2 A a, the center S.

264. *Find the internal fall in the ellipse, the force in the focus.*



$$\frac{\text{V}^2 \text{ down P x}}{\text{V}^2 \text{ in the circle S x}} = \frac{\text{P x}}{\frac{\text{S P}}{2}}$$

$$\frac{\text{V}^2 \text{ in the circle S x}}{\text{V}^2 \text{ in the circle S P}} = \frac{\text{S P}}{\text{S x}}, \text{ force } \propto \frac{1}{\text{distance}^2}$$

$$\frac{V^2 \text{ in the circle S P}}{V^2 \text{ in the ellipse at P}} = \frac{\text{A a}}{2 \text{ H P}}$$

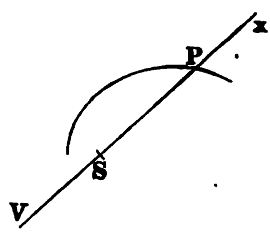
$$\therefore \frac{V^2 \text{ down P x}}{V^2 \text{ in the ellipse at P}} = \frac{\text{P x . A a}}{\text{S x . H P}}$$

$$\therefore \frac{\text{P x}}{\text{S x}} = \frac{\text{H P}}{\text{A a}}$$

$$\therefore \frac{\text{P x}}{\text{S P}} = \frac{\text{H P}}{\text{A a} + \text{H P}}$$

Describe a circle from H with the radius A a. Produce P H to the circumference in F. Join F S. Draw H x parallel to F S.

265. *Generally.*



For *external* falls.

$$\frac{V^2 \text{ down P x}}{V^2 \text{ in the circle distance S P}} = \frac{2\,g\,.\,\text{area A B F D}}{g\,\text{F . S P}} \quad \begin{array}{l}\text{Newton's fig.}\\ \text{F} = \text{force at distance S P}\end{array}$$

$$\frac{V^2 \text{ in the circle S P}}{V^2 \text{ in the curve at P}} = \frac{2 \text{ S P}}{\text{P V}}$$

$$\therefore \frac{V^2 \text{ down P x}}{V^2 \text{ in the curve}} = \frac{4\,.\,\text{A B F D}}{\text{F . P V}}$$

$$\therefore 4\,.\,\text{A B F D} = \text{F . P V}$$

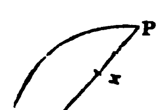
$$\text{Find the area in general} \left\{ \begin{array}{l} \text{ordinate} = \text{F} \\ \text{abscissa} = \text{space} \end{array} \right\}$$

In the general expression make the distance from the center = S P, and a the original height, S x will be found.

266. For *internal* falls.

$$\frac{V^2 \text{ down P x}}{V^2 \text{ in the circle S P}} = \frac{2\,g\,.\,\text{A B F D}}{2\,g\,\text{F . S P}} \quad \begin{array}{l}\text{Newton's fig.}\\ \text{F} = \text{force at P}\end{array}$$

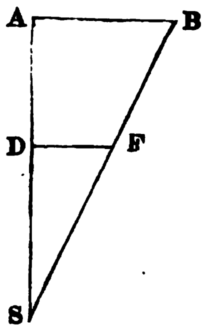
$$\frac{V^2 \text{ in the circle S P}}{V^2 \text{ in the curve at P}} = \frac{2 \text{ S P}}{\text{P V}}$$

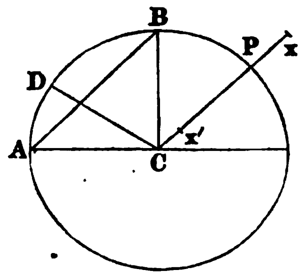


$$\therefore \frac{V^2 \text{ down P x}}{V^2 \text{in the curve at P}} = \frac{4 \text{ A B F D}}{\text{F . P V}}$$

$\therefore$ if the velocities are equal, 4 A B F D = F . P V.

267. Ex. For internal and external falls.





In the ellipse the force tending to the center.

In this case, $DF \propto DS$. Take AB for the force at A. Join BS. $\therefore DF$ is the force at D, and the area $ABFD = \frac{AD}{2}(AB + DF)$ $= \overline{AS - SD}.\overline{AB + DF}$. Let μ equal the absolute force at the distance 1. Let $SA = a$, $SD = x$, $\therefore AB = a\mu$

$$DF = x\mu$$

$$\therefore ABFD = \mu.\frac{\overline{a - x}.\overline{a + x}}{2} = \mu.\frac{a^2 - x^2}{2}$$

and

$$4\,ABFD = F.PV,$$

or

$$a^2 - x^2 = CP.\frac{CD^2}{CP} \text{ in the ellipse,}$$

or

$$a^2 - x^2 = CD^2.$$

For the *external* fall, make $x = CP$, then $a = Cx$, and $Cx^2 - CP^2 = CD^2$,

or
$$Cx^2 = CP^2 + CD^2$$
$$= AC^2 + BC^2$$
$$= AB^2$$
$$\therefore Cx = AB.$$

For the *internal* fall, make $a = CP$, then $x = Cx'$, and

$$CP^2 - Cx'^2 = CD^2,$$

or

$$Cx'^2 = CP^2 - CD^2,$$
$$\therefore Cx' = \sqrt{CP^2 - CD^2}.$$

268. Similarly, in all cases where the velocity in the curve is quadrable, without the Integral Calculus we may find internal and external falls. But generally the process must be by that method.

Thus in the above Ex.

$$v\,d\,v = -g\,\mu\,x\,.\,d\,x$$

$$\therefore v^2 = g\,\mu\,(a^2 - x^2)$$

$$\therefore A\,B\,F\,D = \frac{V^2}{2\,g} = \mu\,\frac{\overline{a^2 - x^2}}{2},$$ as above, &c.

269. And in general,

$$v^2 = \frac{2\,g\,\mu}{n+1}\,(a^{n+1} - x^{n+1}),$$ if the force $\propto x^n$.

Also

$$v^2 = \frac{g}{2}\,F\,.\,P\,V = \frac{g}{2}\,\mu\,\varrho^n\,.\,\frac{2\,p\,d\,\varrho}{d\,p}$$

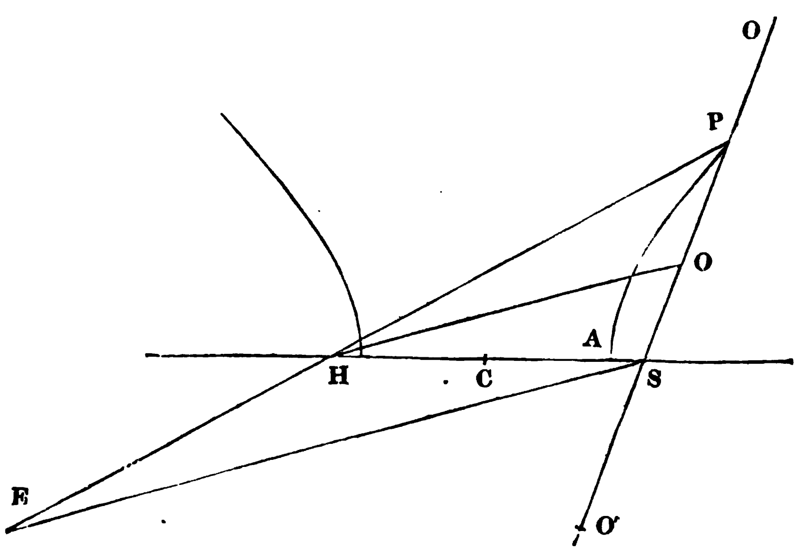
$$\therefore \frac{2\,g\,\mu}{n+1}\,(a^{n+1} - x^{n+1}) = g\,\mu\,\varrho^n\,.\,\frac{p\,d\,\varrho}{d\,p}$$

$$\therefore \frac{2}{n+1}\,(a^{n+1} - x^{n+1}) = \varrho^n\,.\,p\,.\,\frac{d\,\varrho}{d\,p}$$

And to find the external fall, make $x = \varrho$, and from the equation find a, the distance required.

And to find the internal fall make $a = r$, and from the equation find x, the distance required.

270. *Find the external fall in the hyperbola, the force* $\propto \frac{1}{D^2}$ *from the focus.*



V^2 down O P : V^2 in the circle rad. S P : : O P : $\frac{S\,O}{2}$

V^2 in the circle S P : V^2 in the hyperbola at P : : A C : H P

$\therefore$ V² down O P : V² in the hyperbola : : A C . O P : $\frac{SO.HP}{2}$

$$\therefore 2\,AC.OP = SO.HP$$
$$\therefore 2\,AC.SO - 2\,AC.SP = SO.HP$$
$$\therefore SO = -\frac{2\,AC.SP}{HP - 2AC} = -2\,AC$$

To find what this denotes, find the actual velocity in the hyperbola.
Let the force $= \beta$, at a distance $= r$, $\therefore$ the force at the distance $= \frac{\beta . r^2}{x^2}$.

Also

$$\frac{V^2 \text{ in the circle SP}}{2g} = \frac{\beta . r^2}{x^2} \cdot \frac{x}{2} = \frac{\beta x^2}{2x}$$
$$\therefore \frac{V^2 \text{ in the hyperbola}}{2g} = \frac{(2a + x)\beta r^2}{a . 2x}$$
$$= \frac{\beta r^2}{x} + \frac{\beta r^2}{2a}$$

But $\frac{V^2}{2g}$ when the body has been projected from $\infty = \frac{\beta r^2}{x} + \frac{V^2}{2g}$ of projection from ∞, $\therefore \frac{V^2}{2g}$ of projection from $\infty = \frac{\beta . r^2}{2a} = \frac{V^2}{2g}$ down 2 a, F being constant and $= \frac{\beta r^2}{4a^2}$, or $=$ V² from ∞ to O′, when S O′ $=$ 2 A C.

$\therefore$ V in the hyperbola is such as would be acquired by the body ascending from the distance x to ∞ by the action of force considered as repulsive, and then being projected from ∞ back to O′, S O being $=$ 2 A C.

In the opposite hyperbola the velocity is found in the same way, the force repulsive, p externally $= \frac{2\,HC.SP}{2\,AC - HP}$.

271. *Internal fall.*

V² down P O : V² in the circle rad. S O : : P O : $\frac{SP}{2}$

V² in the circle S O : V² in the circle S P : : S P : S O

V² in the circle S P : V² in the hyperbola at P : : A C : H P

$\therefore$ V² down P O : V² in the hyperbola : : A C . P O : $\frac{SO.HP}{2}$

$$\therefore 2\,AC.PO = SO.HP$$

or

$$2\,AC\,(SP - SO) = SO.HP$$
$$\therefore SO = \frac{2\,AC.SP}{2\,AC + HP},$$

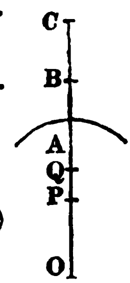
and

$$PO = SP - SO = \frac{SP.HP}{2AC + HP}.$$

Hence make H E = 2 A C, join S E, and draw H O parallel to S E.

Hence the external and internal falls are found, by making V acquired down a certain space p with α^{ble} force equal that down $\frac{1}{4}$. P V by a constant force, P V being known from the curve.

272. *Find how far the body must fall externally to the circumference to acquire* V *in the circle,* F α *distance towards the center of the circle.*



Let O C = p, O B = x, O A = a, C being the point required from which a body falls.

Let the force at A = 1, $\therefore$ the force at B $= \frac{x}{a}$

$v\,d\,v = -g.F.dx$, (for the velocity increases as x decreases)

$$= -g\frac{x}{a}.dx$$

$$\therefore v^2 = \frac{g}{a}.x^2 + C$$

and when v = 0, x = p,

$$\therefore C = \frac{g}{a}.p^2$$

$$\therefore v^2 = \frac{g}{a}(p^2 - x^2)$$

and when x = a,

$$v^2 \text{ at } A = \frac{g}{a}(p^2 - a^2).$$

But

$$v^2 \text{ at } A = 2g.\frac{a}{2}$$

the force at A being constant, and

$$\frac{a}{2} = \frac{PV}{4},$$

$$= g\,a$$

$$\therefore p^2 - a^2 = a^2, \therefore p^2 = 2a^2, \therefore p = \sqrt{2}.a.$$

273. *Find how far the body must fall internally from the circumference to acquire* V *in the circle,* F α *distance towards the center of the circle.*

Let P be the point to which the body must fall, O A = a, O P=p, O Q = x, F at A = 1, $\therefore$ the force at Q $= \frac{x}{a}$.

$$\therefore v\,dv = -g \cdot \frac{x}{a} \cdot dx$$

$$\therefore v^2 = -\frac{g}{a} \cdot x^2 + C,$$

and when $v = 0$, $x = a$,

$$\therefore C = \frac{g}{a} \cdot a^2$$

$$\therefore v^2 = \frac{g}{a}(a^2 - x^2)$$

and when $x = p$,

$$v^2 = \frac{g}{a}(a^2 - p^2) \text{ from } \propto^{\text{ble}} \text{ force}$$

and

$$v^2 = g \cdot a, \text{ from the constant force 1 at } A.$$

$\therefore a^2 - p^2 = a^2$, $\therefore p = 0$, $\therefore$ the body falls from the circumference to the center.

274. Similarly, when $F \propto \frac{1}{\text{distance}}$.

$$O\,C, \text{ or } p \text{ externally} = a\sqrt{e}, \text{ (e = base of hyp. log.)}$$

and

$$O\,P, \text{ or } p \text{ internally} = \frac{a}{\sqrt{e}}.$$

275. When $F \propto \frac{1}{\text{distance}^2}$.

$$p \text{ externally} = 2\,a$$

$$p \text{ internally} = \frac{2\,a}{3}.$$

276. When $F \propto \frac{1}{\text{distance}^3}$.

$$p \text{ externally} = \infty.$$

$$p \text{ internally} = \frac{a}{\sqrt{2}}.$$

277. When $F \propto \frac{1}{\text{distance}^{n+1}}$.

$$p \text{ externally} = a\sqrt[n]{\frac{2}{2-n}}$$

$$p \text{ internally} = a\sqrt[n]{\frac{2}{2+n}}$$

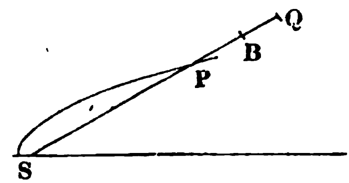
If the force be repulsive, the velocity increases as the distance increases,

$$\therefore v\,dv = g\,F \cdot dx$$

278. *Find how far a body must fall externally to any point* P *in the parabola, to acquire* v *in the curve.* $F \propto \frac{1}{D^2}$, *towards the focus.*

$PV = 4SP = c$, $SQ = p$, $SB = x$, $SP = a$, force at $P = 1$,

$$\therefore F \text{ at } B = \frac{a^2}{x^2},$$



$$\therefore v\,dv = -g\frac{a^2}{x^2}.dx$$

$$\therefore \frac{v^2}{2} = \frac{ga^2}{x} + C,$$

when $v = 0$, $x = p$

$$\therefore C = \frac{ga^2}{p},$$

$$\therefore v^2 = 2ga^2\left(\frac{1}{x} - \frac{1}{p}\right) = 2ga^2\left(\frac{1}{a} - \frac{1}{p}\right) \text{ at P},$$

but

$$v^2 = 2g.\frac{c}{4} = 2ga,$$

$$\therefore \frac{1}{a} - \frac{1}{p} = \frac{1}{a},$$

$$\therefore \frac{1}{p} = 0, \therefore p = \infty.$$

279. Similarly, internally, $p = \frac{a}{2}$.

280. In the ellipse, $F \propto \frac{1}{D^2}$ towards a focus

p externally $= \overline{PH + PS}$. ($\therefore$ describe a circle with the center S, rad. $= 2AC$)

p internally $= \frac{PH.PS}{2AC + PH}$.

(Hence V at P = V in the circle e. d.)

281. In the hyperbola, $F \propto \frac{1}{D^2}$ towards focus

p externally $= -2AC$ (Hence V at P = V in the circle e. d.)

p internally $= \frac{PH.PS}{2AC + PH}$. (Hence V at P = V in the circle e. d., p. 190)

282. In the ellipse $F \propto D$ from the center

p externally $= \sqrt{AC^2 + BC^2}$, (= A B)} (Hence construction)

or ($= \sqrt{CD^2 + CP^2}$)

(Hence also V at P = V in the circle radius C P, when C D = C P)

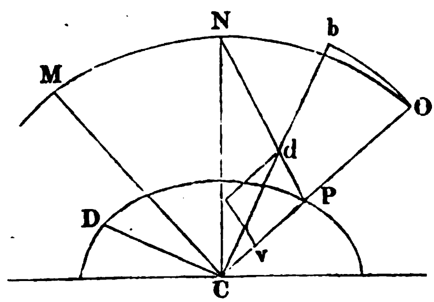
p internally $= \sqrt{CP^2 - CD^2}$.

(Hence if C P = C D, p = 0, and V at P = V in the circle e. d., as was deduced before)

(If C P < C D, p impossible, ∴ the body cannot fall from any distance to C and thus acquire the V in the curve)

283. In the ellipse, F α D from the center.

External fall.



The velocity-curve is a straight line, (since D F α C D, also since F = 0, when C P = 0, this straight line comes to C, as C d b, V α $\sqrt{\text{C O b}}$ α C O, O being the point fallen from, to acquire V at P.

∴ V from O to C : V from P to C : : O C : P C

Also since v d v = — g F . d x, and if the force at the distance 1 = 1, the force at x = x. ∴ v d v = — g x d x, and integrating and correcting, $v^2 = g(p^2 - x^2)$, where p = the distance fallen from.

∴ v α $\sqrt{p^2 - x^2}$, and if a circle be described, with center C, rad. C O α P N (the right sine of the arc whose versed P O is the space fallen through).

∴ V from O to P : V from O to C : : P N : (C M =) O C

and

V from P to C : V in the circle rad. C P : : 1 : 1

(for if P v = $\frac{1}{2}$ P C, v d = C d P) and

V in the circle C P : V in the ellipse : : C P : C D.

Compounding the 4 ratios,

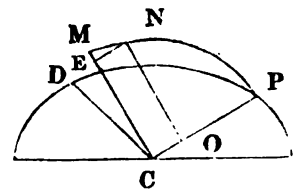
V down O P : V in the ellipse : : P N : C D

∴ Take P N = C D, and

V down O P = V in the ellipse,

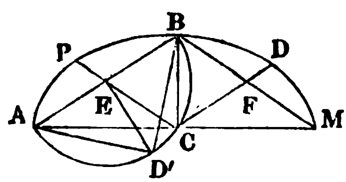
∴ C O = C N = $\sqrt{\text{C P}^2 + \text{C D}^2}$.

Internal fall.



V in the ellipse : V in the circle rad. C P : : C D : C P
V in the circle : V down C P : : 1 : 1
V down C P : V down P O : : (C M =) C P : O N
∴ V in the ellipse : V down P O : : C D : O N
∴ Take O N = C D, and V in the curve = V down P O, and C O $= \sqrt{CP^2 - CD^2}$.

284. *Find the point in the ellipse, the force in the center, where* V = *the velocity in the circle*, e. d.



In this case C P = C D, whence the construction.

Join. A B, describe $\frac{\text{circle}}{2}$ on it, bisect the circumference in D′, join B D′, A D′. From C with A D′ cut the ellipse in P.

$$2\,AD'^2 (= 2\,PC^2) = AB^2 = AC^2 + BC^2 (= CP^2 + CD^2)$$
$$\therefore 2\,CP^2 = CP^2 + CD^2$$
$$\therefore CP^2 = CD^2.$$ (C P will pass through E.)

A simpler construction is to bisect A B in E, B M in F, then C P is the diameter to the ordinate A B, and from the triangles C E B, C F B, C E is parallel to A B, ∴ C D′ is a conjugate to C P and = C P.

285. In the hyperbola, force repulsive, ∝ D, from the center $\left\{\begin{array}{l}\text{p externally (to which body must rise from P,)} = \sqrt{CD^2 + CP^2} \\ \text{p internally (to which body must rise from the center)} = \sqrt{CP^2 - CD^2}\end{array}\right.$

(Hence if the hyperbola be rectangular p internally = 0, or the body must rise through C P.)

286. *In any curve,* F $\propto \frac{1}{D^{n+1}}$, *find* p *externally.*

$$p = a . \left(\frac{a}{a - \frac{n c}{4}}\right)^{\frac{1}{n}},$$

where a = S P, c = P V.

287. If the curve be a logarithmic spiral, c = 2 a,

$$\therefore p = a \left(\frac{a}{a - \frac{n a}{2}}\right)^{\frac{1}{n}}$$

$$= a \left(\frac{1}{1 - \frac{n}{2}}\right)^{\frac{1}{n}}$$

$$\left.\begin{array}{l}\text{also } F \propto \frac{1}{D^3}, \\ \therefore n = 2\end{array}\right\} \therefore p = a \left(\frac{1}{1 - 1}\right)^{\frac{1}{n}} = \infty .$$

288. *In any curve,* F $\propto \frac{1}{D^{n+1}}$, *find* p *internally.*

$$p = a \left(\frac{a}{a + \frac{n c}{4}}\right)^{\frac{1}{2}} . \left(p^n = \frac{4 a^{n+1}}{4 a + n c}\right)$$

289. If the curve be a logarithmic spiral, c = 2 a, n = 2,

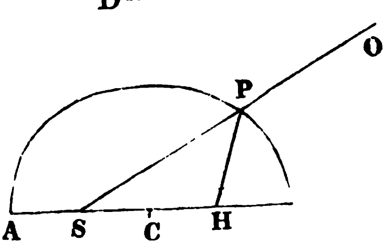
$$\therefore p = a \left(\frac{a}{a + a}\right)^{\frac{1}{n}} = \frac{a}{\sqrt{2}}.$$

290. If the curve be a circle, F in the circumference, c = a, and n = 4,

$$\therefore p \text{ externally} = a \left(\frac{a}{a - a}\right)^{\frac{1}{4}} = \infty$$

$$\text{and } p \text{ internally} = a \left(\frac{a}{a + a}\right)^{\frac{1}{4}} = \frac{a}{\sqrt[4]{2}}.$$

291. *In the ellipse,* F $\propto \frac{1}{D^2}$ *from focus. External fall.*



V^2 down O P : V^2 in the circle radius S P : : O P : $\frac{S O}{2}$, Sect. VII.

V^2 in the circle S P : V^2 in the ellipse at P : : A C : H P,

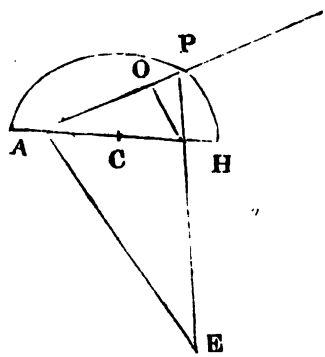
$$\therefore V^2 \text{ down } OP : V^2 \text{ in the ellipse} :: AC.OP : \frac{SO.HP}{2}$$

$$\therefore 2AC.OP = SO.HP$$

$$\therefore SO = \frac{2AC.OP}{HP} = \frac{2AC.SO - 2AC.SP}{HP}$$

$$\therefore SO = \frac{2AC.SP}{2AC - HP} = 2AC.$$

Internal fall.



$$V^2 \text{ down } PO : V^2 \text{ in the circle radius } SO :: PO : \frac{SP}{2},$$

$$V^2 \text{ in the circle } SO : V^2 \text{ in the circle } SP :: SP : SO$$

$$V^2 \text{ in the circle } SP : V^2 \text{ in the ellipse at } P :: AC : HP$$

$$\therefore V^2 \text{ down } PO : V^2 \text{ in the ellipse} :: PO.AC : \frac{SO.HP}{2}$$

$$\therefore 2PO.AC = SO.HP$$

$$\therefore 2SP.AC - 2SO.AC = SO.HP$$

$$\therefore SO = \frac{2AC.SP}{2AC + HP}$$

Hence, make $HE = 2AC$, join SE, and draw HO parallel to ES.

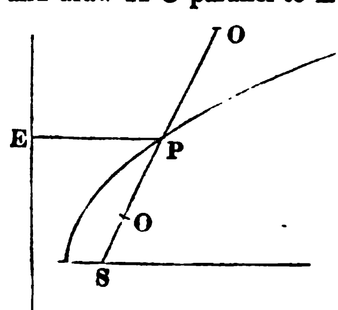
292. *External fall in the parabola,*

$F \propto \frac{1}{D^2}$ from focus.

$V^2 d.OP : V^2$ in the circle radius SP

$:: OP : \frac{SO}{2}$, Sect. VII.

V^2 in the circle $SP : V^2$ in the parabola at $P :: 1 : 2$,

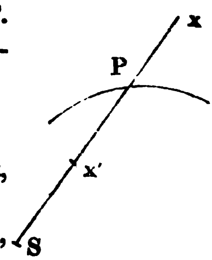


$$\therefore V^2 \text{ down } OP : V^2 \text{ in the parabola} :: OP : SO$$
$$\therefore OP = SO, \therefore SO = \alpha$$

Internal fall.

$$V^2 \text{ down } OP : V^2 \text{ in the circle } SO :: OP : \frac{SP}{2}$$
$$V^2 \text{ in the circle } SO : V^2 \text{ in the circle } SP :: SP : SO$$
$$V^2 \text{ in the circle } SP : V^2 \text{ in the parabola at } P :: 1 : 2$$
$$\therefore V^2 \text{ down } OP : V^2 \text{ in the parabola} :: OP : SO,$$
$$\therefore OP = SO,$$
$$\therefore SO = \frac{SP}{2}.$$

$V = V \text{ down } \frac{PV}{4} = V \text{ down } SP = V.\text{down } EP = V$ of a body describing the parabola by a constant vertical force = force at P.



293. *Find the external fall so that the velocity² acquired* $= n'$. *velocity in the curve,* $F \propto x^n$.

$$v\,dv = -g\mu . x^n . dx, \ (\mu = \text{force distance } 1),$$
$$\therefore v^2 = \frac{2g\mu}{n+1} . (a^{n+1} - x^{n+1}) \ a = \text{original height},$$
$$V^2 \text{ in the curve} = g\,\mu . \frac{p\,d\varrho}{dp} = \frac{g}{2}\mu . c, \text{ if } c = \frac{2p\,d\varrho}{dp},$$
$$\therefore n' . \frac{g}{2}\mu . c = \frac{2g\mu}{n+1} . (a^{n+1} - x^{n+1}), \text{ or } n' . c = \frac{4}{n+1} . (a^{n+1} - x^{n+1})$$

Make $x = SP = \varrho$, and from the equation we get a, which $= Sx$.

For the internal fall, make $a = SP = \varrho$, and from the equation we get x, which $= Sx'$.

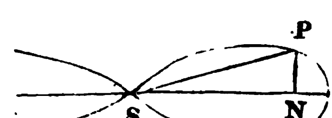
294. *Find the external fall in a* LEMNISCATA.

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2)$$

is a rectangular equation whence we must get a polar one

Let $\angle NSP = \theta$,

$$\therefore y = \varrho . \sin.\theta, \ x = \varrho . \cos.\theta, \ \varrho^2 = (x^2 + y^2)$$
$$\therefore \varrho^4 = a^2 . (\varrho^2 (\cos.^2\theta - \sin.^2\theta)) = a^2 \varrho^2 . \cos. 2\theta,$$
$$\therefore \varrho^2 = a^2 . \cos. 2\theta$$
$$\therefore 2\theta = \angle \left(\cos. = \frac{\varrho^2}{a^2}\right),$$
$$\therefore 2\,d\theta = -\frac{\frac{2\varrho\,d\varrho}{a^2}}{\sqrt{1 - \frac{\varrho^4}{a^4}}} = \frac{-2\varrho\,d\varrho}{\sqrt{a^4 - \varrho^4}}$$



$$\therefore d\,\theta^2 = \frac{\varrho^2\,d\,\varrho^2}{a^4 - \varrho^4},$$

$$\therefore \frac{d\,\varrho^2}{d\,\theta^2} = \frac{a^4 - \varrho^4}{\varrho^2}$$

but in general

$$\varrho\,.\,d\,\theta = \frac{d\,\varrho\,.\,p}{\sqrt{\varrho^2 - p^2}},$$

$$\therefore \varrho^4\,d\,\theta^2 - \varrho^2\,d\,\theta^2\,p^2 = d\,\varrho^2\,.\,p^2,$$

$$\therefore p^2 = \frac{\varrho^4\,d\,\theta^2}{\varrho^2\,d\,\theta^2 + d\,\varrho^2} = \frac{\varrho^4}{\varrho^2 + \frac{d\,\varrho^2}{d\,\theta^2}}$$

$$= \text{in this case } \frac{\varrho^4}{\varrho^2 + \frac{a^4 - \varrho^4}{\varrho^2}}$$

$$\therefore p^2 = \frac{\varrho^6}{a^4},$$

$$\therefore \frac{1}{p^2} = \frac{a^4}{\varrho^6},$$

$$\therefore -\frac{2\,d\,p}{p^3\,d\,\varrho} = -\frac{6\,.\,a^4}{\varrho^7},$$

$$\therefore \text{force to S} \propto \frac{1}{\varrho^7}$$

$$v\,d\,v = -\frac{g\,\mu\,.}{x^7}\,.\,d\,x,$$

$$\therefore v^2 = \frac{2\,g\,\mu}{6}\left(\frac{1}{x^6} - \frac{1}{a^6}\right)$$

Also

$$P\,V = \frac{2\,p\,d\,\varrho}{d\,p} = 2\,p\,.\,\frac{a^2}{3\,\varrho^2} = \frac{2\,.\,\varrho^3}{a^2}\,.\,\frac{a^2}{3\,\varrho^2} = \frac{2\,\varrho}{3},$$

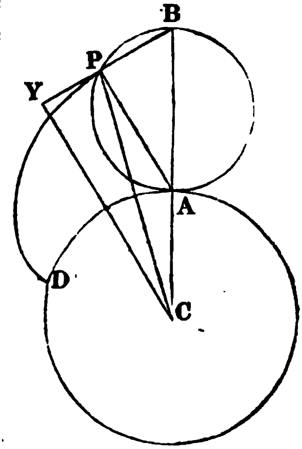
$$\therefore v^2 = \frac{g\,\mu}{\varrho^7}\,.\,\frac{2\,\varrho}{3} = \frac{2\,g\,\mu\,.}{6}\,.\,\frac{1}{\varrho^6},$$

Make x in the formula above $= \varrho$,

$$\therefore \frac{1}{\varrho^6} - \frac{1}{a^6} = \frac{1}{\varrho^6},$$

$$\therefore \frac{1}{a^6} = 0, \therefore a \text{ is infinite.}$$

295. *Find the force and external fall in an* EPICYCLOID



$$CY^2 = CP^2 - YP^2 = CP^2 - CA^2 \cdot \frac{YB^2}{CB^2}$$

Let

$CY = p,\ CP = \rho,\ CB = c,\ CA = b,$

$$\therefore p^2 = \rho^2 - b^2 \cdot \frac{c^2 - p^2}{c^2}$$

$$\therefore c^2 p^2 = c^2 \rho^2 - b^2 c^2 + b^2 p^2$$

$$\therefore p^2 = \frac{c^2(\rho^2 - b^2)}{c^2 - b^2}$$

$$\therefore \frac{1}{p^2} = \frac{c^2 - b^2}{c^2(\rho^2 - b^2)},$$

$$\therefore -\frac{2\,dp}{p^3} = \frac{c^2 - b^2}{c^2}\left(\frac{-2\,d\rho \cdot \rho}{(\rho^2 - b^2)^2}\right)$$

$$\therefore \text{force} \propto \frac{\rho}{(\rho^2 - b^2)^2} \propto \frac{\rho}{p^4}$$

(as in the Involute of the circle which is an Epicycloid, when the radius of the rota becomes infinite.)

Having got $\propto^n$ of force, we can easily get the external (or internal) fall.

296. *Find in what cases we can integrate for the Velocity and Time.*

Case 1. Let force $\propto x^n$,

$$\therefore v\,dv = g\mu \cdot x^n\,dx,$$

$$\therefore v^2 = \frac{2g\mu}{n+1}(a^{n+1} - x^{n+1})$$

$$\therefore t = \int \frac{-dx}{v} = \sqrt{\frac{n+1}{2g\mu}} \cdot \int \frac{-dx}{\sqrt{(a^{n+1} - x^{n+1})}}$$

Now in general we can integrate $x^m\,dx \cdot (a + b x^n)^{\frac{p}{q}}$, when

$\frac{m+1}{n}$ is whole or $\frac{m+1}{n} + \frac{p}{q}$ whole.

$\therefore$ in this case, we can integrate, when $\frac{1}{n+1}$, or $\frac{1}{n+1} - \frac{1}{2}$, is whole.

Let

$$\frac{1}{n+1} = p \text{ any whole number}$$

$$\therefore n + 1 = \frac{1}{p},$$

$$\therefore n = \frac{1-p}{p}, \text{ (p being positive), } (\alpha)$$

Let

$$\frac{1}{n+1} - \frac{1}{2} = p,$$

$$\therefore \frac{1}{n+1} = p + \frac{1}{2} = \frac{2p+1}{2},$$

$$\therefore n + 1 = \frac{2}{2p+1},$$

$$\therefore n = \frac{1-2p}{1+2p}. \quad (\beta)$$

$\therefore$ these formulæ admit only 0 and 1 for integer positive values of n, and no positive fractional values. $\therefore$ we can integrate when $F \propto x$, or $F \propto 1$.

297. Case 2. Let force $\propto \frac{1}{x^n}$,

$$\therefore v\,dv = -g\mu \frac{dx}{x^n},$$

$$\therefore v^2 = \frac{2g\mu}{n-1} \cdot \left(\frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}\right)$$

$$\therefore t = \int \frac{-dx}{v} = \sqrt{\frac{\overline{n-1}.a^{n-1}}{2g\mu}} \cdot \int \frac{-dx \cdot x^{\frac{n-1}{2}}}{\sqrt{a^{n-1} - x^{n-1}}}$$

in which case we can integrate, when $\frac{\frac{n-1}{2}+1}{n-1}$, or $\frac{\frac{n-1}{2}+1-\frac{1}{2}}{n-1}$, whole.

i. e. if $\frac{1}{2} + \frac{1}{n-1}$, or $\frac{1}{n-1}$, be whole.

Let $\frac{1}{n-1} = p$, any whole positive No.,

$$\therefore n - 1 = \frac{1}{p}, \therefore n = \frac{p+1}{p}, \quad (\alpha')$$

Let $\frac{1}{2} + \frac{1}{n-1} = p$,

$$\therefore \frac{1}{n-1} = \frac{2p-1}{2},$$

$$\therefore n - 1 = \frac{2}{2p-1},$$

$$\therefore n = \frac{2p+1}{2p-1}. \quad (\beta')$$

$\therefore$ these formulæ admit any values of n, in which the numerator exceeds the denominator by 1, or in which the numerator and denominator are any two successive odd numbers, the numerator being the greater.

$$\left.\begin{array}{l} \therefore \text{ we can integrate, when } F \propto \frac{1}{x^2}, \frac{1}{x^{\frac{3}{2}}}, \frac{1}{x^{\frac{4}{3}}}, \frac{1}{x^{\frac{5}{4}}}, \text{ \&c.} \\ \text{or} \\ \qquad \frac{1}{x^3}, \frac{1}{x^{\frac{5}{3}}}, \frac{1}{x^{\frac{7}{5}}}, \frac{1}{x^{\frac{9}{7}}}, \text{ \&c.} \end{array}\right\}$$

298. Case 3. The formulæ (α') (β'), in which p is positive, cannot become negative. But the formulæ (α) and (β) may. From which we can integrate, when $F \propto \frac{1}{x^{\frac{1}{2}}}, \frac{1}{x^{\frac{2}{3}}}, \frac{1}{x^{\frac{3}{5}}}, \frac{1}{x^{\frac{4}{3}}}$, &c.

or when $F \propto \frac{1}{x^{\frac{1}{3}}}, \frac{1}{x^{\frac{5}{3}}}, \frac{1}{x^{\frac{4}{7}}}, \frac{1}{x^{\frac{7}{9}}}$, &c.

299. *When the force* $\propto x^n$, *find* $\propto^n$. *of times from different altitudes to the center of force. Find the same, force* $\propto \frac{1}{x^n}$.

$$F \propto x^n, \therefore v\,d\,v = -g\,\mu\,x^n\,d\,x,$$

$$\therefore v^2 = \frac{2\,g\,\mu}{n+1}\cdot(a^{n+1} - x^{n+1})$$

$$\therefore d\,t = -\frac{d\,x}{v} \propto \frac{-d\,x}{\sqrt{a^{n+1} - x^{n+1}}} \text{ which is of } -\frac{n+1}{2} \text{ dimensions,}$$

$$\therefore t \text{ will be of } -\frac{n-1}{2} \text{ dimensions.}$$

and when $x = 0$, t will $\propto \frac{1}{a^{\frac{n-1}{2}}}$.

$$F \propto \frac{1}{x^n}, \therefore t \propto \frac{1}{a^{-\frac{n-1}{2}}} \propto a^{\frac{n+1}{2}}.$$

$$t \propto \int \frac{-d\,x}{\sqrt{a^{n+1} - x^{n+1}}} \propto \frac{1}{a^{\frac{n+1}{2}}} \int \frac{-d\,x}{\sqrt{1 - \left(\frac{x}{a}\right)^{n+1}}}$$

$$\propto \frac{1}{a^{\frac{n+1}{2}}} \int -d\,x \left\{1 - \left(\frac{x}{a}\right)^{n+1}\right\}^{-\frac{1}{2}}$$

$$\propto \frac{1}{a^{\frac{n+1}{2}}} \int -d\,x.\left\{1 + \frac{1}{2}\cdot\left(\frac{x}{a}\right)^{n+1} + \frac{1.3}{2.4}\cdot\left(\frac{x}{a}\right)^{2.\overline{n+1}} + \&c.\right\}$$

$$\propto \frac{1}{a^{\frac{n+1}{2}}}\left(C - \left\{x + \frac{1}{2}\cdot\frac{x^{n+2}}{n+2\,a^{n-1}} + \frac{1.3}{2.4}\cdot\frac{x^{2n+3}}{2n+3.a^{2n-2}} + \&c.\right\}\right)$$

when $t = 0$, $x = a$,

$$\therefore C = \left\{a + \frac{1}{2}\cdot\frac{a}{n+2} + \frac{1.3}{2.4}\cdot\frac{a}{2n+3} + \&c.\right\}$$

$$= a\left\{1 + \frac{1}{2}\cdot\frac{1}{n+2} + \frac{1.3}{2.4}\cdot\frac{1}{2n+3} + \&c.\right\}$$

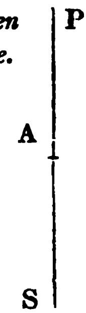
$$\therefore t \propto \frac{1}{a^{\frac{n+1}{2}}}\cdot\left(a.\left\{1 + \frac{1}{2}\cdot\frac{1}{n+2} + \&c.\right\} - \left\{x + \frac{1}{2}\cdot\frac{x^{n+2}}{n+2}\cdot\frac{1}{a^{n-1}} + \&c.\right\}\right)$$

$$\therefore \text{when } x = 0,\ t \propto \frac{a}{a^{\frac{n+1}{2}}} \propto \frac{1}{a^{\frac{n-1}{2}}}$$

$$\text{when n is negative } t \propto \frac{1}{a^{\frac{-n-1}{2}}} \propto a^{\frac{n+1}{2}}.$$

COR. If n be positive and greater than 1, the greater the altitude, the less the time to the center.

300. *A body is projected up* P A *with the velocity* V *from the given point* A, *force in* S $\propto$ x^n, *find the height to which the body will rise.*



$$v\,d\,v = -g\,\mu\,x^n\,d\,x,$$

for the velocity decreases as x increases,

$$\therefore v^2 = \frac{2\,g\,\mu}{n+1}.\,x^{n+1} + C$$

$$\text{when } v = V,\ x = a,$$

$$\therefore C = V^2 + \frac{2\,g\,\mu}{n+1}.\,a^{n+1}$$

$$\therefore \frac{2\,g\,\mu}{n+1}.\,(x^{n+1} - a^{n+1}) = V^2 - v^2$$

$$\text{Let } v = 0,$$

$$\therefore \frac{2\,g\,\mu}{n+1}.\,(x^{n+1} - a^{n+1}) = V^2$$

$$\therefore x^{n+1} - a^{n+1} = \frac{V^2.\overline{n+1}}{2\,g\,\mu}$$

$$\therefore x^{n+1} = \frac{V^2.\overline{n+1} + 2\,g\,\mu.\,a^{\overline{n+1}}}{2\,g\,\mu}$$

$$\therefore x = \left(\frac{V^2.\overline{n+1} + 2\,g\,\mu.\,a^{\overline{n+1}}}{2\,g\,\mu}\right)^{\frac{1}{n+1}}.$$

COR. Let n $= -2$, and V $=$ the velocity down $\frac{a}{2}$, force at A constant, $=$ velocity in the circle distance S A.

$$\therefore x = \left(\frac{-V^2 + \frac{2\,g\,\mu}{a}.}{2\,g\,\mu}\right)^{-1} = \frac{2\,g\,\mu}{\frac{2\,g\,\mu}{a} - V^2}$$

$$= \frac{2\,g\,\mu}{\frac{2\,g\,\mu}{a} - \frac{g\,\mu}{a^2}.\,a} = \frac{2}{\frac{2}{a} - \frac{1}{a}} = 2\,a.$$

SECTION VIII.

301. Prop. XLI. Resolving the centripetal force I N or D E (F) into the tangential one I T (F′) and the perpendicular one T N, we have (46)

$$\mathrm{I\,N : I\,T :: F : F' :: \frac{d\,v}{d\,t} : \frac{d\,v'}{d\,t'}}$$

$$\therefore \mathrm{d\,v : d\,v' :: d\,t \times I\,N : d\,t' \times I\,T.}$$

But since (46)

$$\mathrm{d\,t = \frac{d\,s}{v},\ d\,t' = \frac{d\,s'}{v'}}$$

and by hypothesis

$$\mathrm{v = v'}$$

$$\therefore \mathrm{d\,t : d\,t' :: d\,s : d\,s' :: I\,N : I\,K}$$

$$\therefore \mathrm{d\,v : d\,v' :: I\,N^2 : I\,K \times I\,T}$$

$$:: 1 \quad : 1$$

or

$$\mathrm{d\,v = d\,v'},$$

&c. &c.

OTHERWISE.

302. By 46, we have generally

$$\mathrm{v\,d\,v = g\,F\,d\,s}$$

s being the direction of the force F. Hence if s′ be the straight line and s the trajectory, &c. we have

$$\mathrm{v\,d\,v = g\,F\,d\,s}$$

$$\mathrm{v'\,d\,v' = g\,F'\,d\,s'}$$

$$\therefore \mathrm{v^2 - V^2 = 2\,g \int F\,d\,s}$$

$$\mathrm{v'^2 - V'^2 = 2\,g \int F'\,d\,s'}$$

V and V′ being the given values of v and v′ at given distances by which the integrals are corrected.

Now since the central body is the same at the same distance the central force must be the same in both curve and line. Therefore, resolving F

when at the distance s into the tangential and perpendicular forces, we have

$$F' = F \times \frac{IT}{IN} = F \times \frac{IN}{IK}$$

$$= F \times \frac{ds}{ds'}$$

$$\therefore F' ds' = F ds$$

and

$$v'^2 - V'^2 = 2g\int F ds = v^2 - V^2$$

which shows that if the velocities be the same at any two equal distances, they are equal at all equal distances — i. e. if

$$V = V'$$

then

$$v = v'.$$

303. COR. 2. By Prop. XXXIX,

$$v^2 \propto ABGE.$$

But in the curve

$$y \propto F \propto A^{n-1}$$

$$\therefore y dx \propto A^{n-1} dA$$

Therefore (112)

$$ABGE = \int y dx \propto -\frac{A^n}{n} + C$$

$$\propto \frac{P^n - A^n}{n}$$

Hence

$$v^2 \propto P^n - A^n.$$

OTHERWISE.

304. Generally (46)

$$v dv = -g F ds$$

and if

$$F = \mu s^{n-1}$$

then

$$v^2 = \frac{2g\mu}{n}(C - s^n)$$

But when $v = 0$, let $s = P$; then

$$0 = \frac{2g\mu}{n}(C - P^n)$$

and

$$C = P^n.$$

$$\therefore v^2 = \frac{2 g \mu}{n} (P^n - s^n)$$

in which s is any quantity whatever and may therefore be the radius vector of the Trajectory A; that is

$$v^2 = \frac{2 g \mu}{n} (P^n - A^n) \text{ or } = \frac{2 g \mu}{n} (D^n - \varrho^n)$$

in more convenient notation.

N. B. From this formula may be found the spaces through which a body must fall *externally* to acquire the velocity in the curve (286, &c.)

305. Prop. XLI. *Given the centripetal force to construct the Trajectory, and to find the time of describing any portion of it.*

By Prop. XXXIX,

$$v = \sqrt{2g} \cdot \sqrt{ABFD} = \frac{ds}{dt} (46) = \frac{IK}{dt}$$

But

$$dt = ICK \times \frac{\text{Time}}{\text{Area}} = IC \times KN \times \frac{\text{Time}}{2 \text{ Area}}.$$

$$= \frac{A \times KN}{P \times V}$$ (P being the perpendicular upon the tangent when the velocity is V. See 125, &c.)

Moreover, if V be the velocity at V, by Prop. XXXIX,

$$V = \sqrt{2g} \cdot \sqrt{ABLV}.$$

Whence

$$\sqrt{ABFD} = \frac{P\sqrt{ABLV}}{A} \times \frac{IK}{KN}$$

$\therefore$ putting

$$Z = \frac{P\sqrt{ABLV}}{A} \left(\text{or} = \frac{Q}{A} = \frac{P \times V}{\sqrt{2gA}}\right) \quad . \; . \; (1)$$

we have

$$ABFD : Z^2 :: IK^2 : KN^2$$

$$\therefore ABFD - Z^2 : Z^2 :: IK^2 - KN^2 : KN^2$$

and

$$\sqrt{ABFD - Z^2} : Z = \frac{Q}{A} :: IN : KN$$

$$\therefore A \times KN = \frac{Q \times IN}{\sqrt{(ABFD - Z^2)}} \quad . \; . \; . \; . \; . \; (2)$$

Also since similar triangles are to one another in the duplicate ratio of their homologous sides

$$YX \times XC = A \times KN \times \frac{CX^2}{A^2}$$

$$= \frac{Q \times CX^2 \times IN}{A^2 \sqrt{(ABFD - Z^2)}} \quad . \quad . \quad . \quad (3)$$

and putting

$$y = Db = \frac{Q}{2\sqrt{(ABFD - Z^2)}}$$

and

$$y' = Dc = \frac{Q \times CX^2}{2A^2 \sqrt{(ABFD - Z^2)}}.$$

Then

$$\left.\begin{array}{l}\text{Area } VCI = \int y\, dx = VDba \\ \text{Area } VCX = \int y'\, dx = VDca\end{array}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Now (124)

$$t = \frac{2\,VCI}{P \times V} = \frac{2\,VDba}{P \times V}$$

or

$$= \frac{2\,VDba}{\sqrt{2g} \,.\, P \times \sqrt{ABLV}} \quad . \quad . \quad . \quad (5)$$

the time of describing VI.

Also, if $\angle\, VCI = \theta$, we have

$$VDca = VCX = \frac{XV \times CV}{2} = \frac{\theta \times CV^2}{2}$$

$$\theta = \frac{2\,VDca}{P^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which gives the Trajectory.

306. *To express equations* (5) *and* (6) *in terms of* ϱ *and* θ, ($\varrho = A$).

First

$$ABFD = \frac{v^2}{2g}$$

$$ABLV = \frac{V^2}{2g}$$

and

$$Q = \frac{P \times V}{\sqrt{2g}}$$

$$\therefore Z^2 = \frac{Q^2}{\varrho^2} = \frac{P^2 \times V^2}{2g\varrho^2}$$

$$\therefore ABFD - Z^2 = \frac{v^2}{2g} - \frac{P^2 \times V^2}{2g\varrho^2}$$

$$= \frac{1}{2g\varrho^2} \times (\varrho^2 v^2 - P^2 V^2)$$

Hence

$$y = \frac{P \times V_\varrho}{2\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

and

$$y' = \frac{P^3 \times V}{2\varrho\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

$$\therefore V D b a = \frac{P V}{2} \int \frac{\varrho d \varrho}{\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

and

$$V D c a = \frac{P^3 V}{2} \times \int \frac{d \varrho}{\varrho \sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

$$\therefore t = \int \frac{\varrho d \varrho}{\sqrt{(\varrho^2 v^2 - P^2 V^2)}}$$

But by Prop. XL.

$$v^2 = 2 \int g F d \varrho$$

the integral being taken from $v = 0$, or from $\varrho = D$, D being the same as P in 304.

$$\therefore t = \int \frac{\int \varrho d \varrho}{\sqrt{(2 \varrho^2 \int g F d \varrho - P^2 V^2)}}, \text{ or } = \int \frac{\int \varrho d \varrho}{\sqrt{(\varrho^2 v^2 - P^2 V^2)}} \quad . . (7)$$

$$\theta = \int \frac{P \times V d \varrho}{\varrho \sqrt{(-2 \varrho^2 \int g F d \varrho - P^2 V^2)}}, \text{ or} = \int \frac{P V d \varrho}{\varrho \sqrt{(\varrho^2 v^2 - P^2 V^2)}} \quad . (8)$$

307. *To find* t *and* θ *in terms of* ϱ *and* p.

Since (125)

$$v^2 = \frac{P^2 V^2}{p^2} = -2 \int g F d \varrho$$

$$\therefore t = \int \frac{\int p \varrho d \varrho}{\sqrt{(P^2 V^2 \varrho^2 - P^2 V^2 p^2)}}$$

$$= \frac{1}{P V} \times \int \frac{\varrho d \varrho}{\sqrt{\left(\frac{\varrho^2}{p^2} - 1\right)}} \quad . . . . (9)$$

and

$$\theta = \int \frac{d \varrho}{\varrho \sqrt{\left(\frac{\varrho^2}{p^2} - 1\right)}} \quad . . . . (10)$$

But previous to using these forms we must find the equation to the trajectory, thus (139)

$$\frac{P^2 V^2}{g} \times \frac{d p}{p^3 d \varrho} = F = f(\varrho)$$

f denoting the law of force.

$$\therefore \frac{P^2 V^2}{2 g} \times \left(\frac{1}{P^2} - \frac{1}{p^2}\right) = \int d \varrho f \varrho$$

or

$$p^2 = \frac{P^2 V^2}{V^2 - 2g\int d\varrho f\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

308. To these different methods the following are examples:

1st. Let F α $\varrho = \mu\varrho$. Then (see 304)

$$\therefore v^2 = g\mu (D^2 - \varrho^2)$$

and if P and V belong to an apse or when $P = \varrho$;

$$V^2 = g\mu (D^2 - P^2)$$

$$\therefore t = \frac{1}{\sqrt{g\mu}} \times \int \frac{\varrho d\varrho}{\sqrt{\{\varrho^2(D^2-\varrho^2) - P^2(D^2-P^2)\}}}.$$

Let $\varrho^2 - \frac{D^2}{2} = u$. Then we easily get

$$2t\sqrt{g\mu} = \int \frac{du}{\sqrt{\left\{\left(P^2 - \frac{D^2}{2}\right)^2 - u^2\right\}}}$$

$$= \sin.^{-1} \frac{u}{P^2 - \frac{D^2}{2}} + C$$

and making $t = 0$ at an apse or when $\varrho = P$, we find

$$C = -\sin.^{-1}. \frac{P^2 - \frac{D^2}{2}}{P^2 - \frac{D^2}{2}} = -\sin.^{-1} 1$$

$$= -\frac{\pi}{2}.$$

$$\therefore t = \frac{1}{2\sqrt{.g\mu}} \times \left\{\sin.^{-1} \frac{\varrho^2 - \frac{D^2}{2}}{P^2 - \frac{D^2}{2}} - \frac{\pi}{2}\right\} \quad . \quad . \quad . \quad (1)$$

Also

$$\frac{\theta}{PV} = \int \frac{dt}{\varrho^2} = \frac{1}{2\sqrt{g\mu}} \int \frac{du}{\left(u + \frac{D^2}{2}\right)\sqrt{\left\{\left(P^2 - \frac{D^2}{2}\right)^2 - u^2\right\}}}$$

and assuming

$$P^2 - \frac{D^2}{2} - u = v^2 \times \left(P^2 - \frac{D^2}{2} + u\right)$$

we get

$$\frac{\theta}{PV} = \frac{1}{2\sqrt{g\mu}P.\sqrt{D^2 - P^2}} \times \left\{\sin.^{-1} \frac{\left(P^2 - \frac{D^2}{2}\right)^2 + \frac{D^2}{2}\left(\varrho^2 - \frac{D^2}{2}\right)}{\varrho^2\left(P^2 - \frac{D^2}{2}\right)} + C\right\}$$

and making $\theta = 0$, when $\varrho = P$ we find

$$C = -\sin.^{-1} 1 = -\frac{\pi}{2}.$$

Also

$$V = \sqrt{g\mu}.\sqrt{(D^2 - P^2)}$$

$$\therefore 2\theta + \frac{\pi}{2} = \sin.^{-1} \frac{\left(P^2 - \frac{D^2}{2}\right)^2 + \frac{D^2}{2}\left(\varrho^2 - \frac{D^2}{2}\right)}{\varrho^2\left(P^2 - \frac{D^2}{2}\right)}$$

$$\therefore \frac{\left(P^2 - \frac{D^2}{2}\right)^2 + \frac{D^2}{2}\left(\varrho^2 - \frac{D^2}{2}\right)}{\varrho^2\left(P^2 - \frac{D^2}{2}\right)} = \sin.\left(2\theta + \frac{\pi}{2}\right)$$

$$= \cos. 2\theta = 2\cos.^2\theta - 1$$

which gives

$$\varrho^2 = \frac{P^2(D^2 - P^2)}{P^2 - (2P^2 - D^2)\cos.^2\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Now the equation to the ellipse, ϱ and θ being referred to its center, is

$$\varrho^2 = \frac{b^2}{1 - e^2\cos.^2\theta}.$$

Therefore the trajectory is an ellipse the center of force being in its center, and we have its semiaxes from

$$b^2 = D^2 - P^2$$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{2P^2 - D^2}{P^2}$$

viz.

$$\left.\begin{array}{l} b = \sqrt{(D^2 - P^2)} \\ \text{and} \quad a = P \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

which latter value was already assumed.

To find the Periodic time.

From (3) it appears that when

$$t = \frac{T}{4}, \text{ or } \theta = \frac{\pi}{2}, \varrho = b = \sqrt{(D^2 - P^2)}$$

and substituting in (1) we have

$$\frac{T}{4} = \frac{1}{2\sqrt{g\mu}} \times \left\{\sin.^{-1}\frac{\frac{D^2}{2} - P^2}{P^2 - \frac{D^2}{2}} - \frac{\pi}{2}\right\}$$

$$= \frac{1}{2\sqrt{g\mu}} \times \left\{\sin.^{-1}(-1) - \frac{\pi}{2}\right\}$$

But

$$\sin.^{-1}(-1) = \frac{3\pi}{2}.$$

$$\therefore \frac{T}{4} = \frac{\pi}{2\sqrt{g\mu}}$$

and

$$T = \frac{2\pi}{\sqrt{g\mu}} \quad \dots\dots\dots\dots \quad (4)$$

which has already been found otherwise (see 147).

To apply (9) and (10) of 307 to this example we must first integrate (11) where $f\varrho = \mu\varrho$; that is since

$$\int \mu\varrho\, d\varrho = \frac{\mu}{2}\varrho^2 + C$$

$$= \frac{\mu}{2}\varrho^2 - \frac{\mu}{2}P^2$$

we have

$$p^2 = \frac{P^2 V^2}{V^2 - \mu g \varrho^2 + \mu g P^2}.$$

But

$$V^2 = g\mu(D^2 - P^2)$$

$$\therefore p^2 = \frac{P^2.(D^2 - P^2)}{D^2 - \varrho^2} \quad \dots\dots\dots\dots \quad (5)$$

which also indicates an ellipse referred to its center, the equation being generally

$$p^2 = \frac{a^2 b^2}{a^2 + b^2 - \varrho^2}.$$

Hence

$$\frac{\varrho^2}{p^2} - 1 = \frac{\varrho^2(D^2 - \varrho^2) - P^2(D^2 - P^2)}{P^2(D^2 - P^2)}$$

$$\therefore t = \frac{1}{\sqrt{g\mu}} \int \frac{\varrho\, d\varrho}{\sqrt{\{\varrho^2(D^2 - \varrho^2) - P^2(D^2 - P^2)\}}}$$

the same as before.

With regard to θ, the axes of the ellipse being known from (5) we have the polar equation, viz.

$$\varrho^2 = \frac{b^2}{1 - e^2 \cos.^2 \theta}.$$

309. Ex. 2. Let $F = \frac{\mu}{\varrho^2}$. Then (304)

$$v^2 = \frac{2g\mu}{-1} \times (D^{-1} - \varrho^{-1})$$

$$= 2 g \mu \times \frac{D - \rho}{D \rho}$$

and

$$V^2 = 2 g \mu \times \frac{D - P}{D P}$$

P and V belonging to an apse.

$$\therefore t = \int \frac{\sqrt{D}}{\sqrt{2 g \mu}} \times \frac{\rho d \rho}{\sqrt{(D \rho - \rho^2 - D P + P^2)}}$$

which, adding and subtracting $\frac{D^2}{4}$, transforms to

$$t = \frac{\sqrt{D}}{\sqrt{2 g \mu}} \times \int \frac{\rho d \rho}{\sqrt{\left\{\left(P - \frac{D}{2}\right)^2 - \left(\rho - \frac{D}{2}\right)^2\right\}}}$$

and making $\rho - \frac{D}{2} = u$

$$t = \frac{\sqrt{D}}{\sqrt{2 g \mu}} \times \int \frac{\left(u + \frac{D}{2}\right) d u}{\sqrt{\left\{\left(P - \frac{D}{2}\right)^2 - u^2\right\}}}$$

$$= \sqrt{\frac{D}{2 g \mu}} \times \left\{C - \sqrt{\left\{\left(P - \frac{D}{2}\right)^2 - u^2\right\}} + \frac{D}{2} \sin.^{-1} \frac{u}{P - \frac{D}{2}}\right\}$$

(see 86).

Let $t = 0$, when $\rho = P$. Then

$$C = -\frac{D}{2} \sin.^{-1} 1 = -\frac{D}{2} \times \frac{\pi}{2}.$$

$$\therefore t = \sqrt{\frac{D}{2 g \mu}} \times \left\{\frac{D}{2}\left(\sin.^{-1} \frac{\rho - \frac{D}{2}}{P - \frac{D}{2}} - \frac{\pi}{2}\right) - \sqrt{\left\{\left(P - \frac{D}{2}\right)^2 - \left(\rho - \frac{D}{2}\right)^2\right\}}\right\} \quad (1)$$

Also

$$\frac{\theta}{P V} = \int \frac{d t}{\rho^2} = \sqrt{\frac{D}{2 g \mu}} \times \int \frac{d u}{\left(u + \frac{D}{2}\right) \sqrt{\left\{\left(P - \frac{D}{2}\right)^2 - u^2\right\}}}.$$

But assuming

$$P - \frac{D}{2} - u = v^2 \times \left(P - \frac{D}{2} + u\right)$$

the above becomes rationalized, and we readily find

$$\int \frac{du}{\left(u+\frac{D}{2}\right)\sqrt{\left\{\left(P-\frac{D}{2}\right)^2-u^2\right\}}} =$$

$$\frac{1}{\sqrt{P.(D-P)}} \times \left\{\tan.^{-1} \frac{\left(P-\frac{D}{2}\right)^2+\frac{D}{2}u}{\sqrt{P.(D-P)} \times \sqrt{\left\{\left(P-\frac{D}{2}\right)^2-u^2\right\}}} + C\right\}$$

and making $\theta = 0$, when $\varrho = P$, or when $u = P - \frac{D}{2}$, we get

$$C = -\tan.^{-1}\frac{1}{0} = -\frac{\pi}{2}.$$

Hence, since moreover

$$V = \sqrt{2g\mu}.\sqrt{\frac{D-P}{DP}}$$

$$\theta + \frac{\pi}{2} = \tan.^{-1} \frac{\left(P-\frac{D}{2}\right)^2+\frac{D}{2}\left(\varrho-\frac{D}{2}\right)}{\sqrt{P.(D-P)} \times \sqrt{\left\{\left(P-\frac{D}{2}\right)^2-\left(\varrho-\frac{D}{2}\right)^2\right\}}}$$

or

$$= \sin.^{-} \frac{\left(P-\frac{D}{2}\right)^2-\frac{D}{2}\left(\varrho-\frac{D}{2}\right)}{\left(P-\frac{D}{2}\right)\varrho}$$

$$\therefore \frac{\left(P-\frac{D}{2}\right)^2+\frac{D}{2}.\left(\varrho-\frac{D}{2}\right)}{\left(P-\frac{D}{2}\right)\varrho} = \sin.\left(\theta+\frac{\pi}{2}\right) = \cos.\theta$$

$$\therefore \varrho = \frac{P^2-PD}{-\frac{D}{2}+\left(P-\frac{D}{2}\right)\cos.\theta}$$

$$= \frac{2P.(D-P)}{D} \times \frac{1}{1+\left(1-\frac{2P}{D}\right)\cos.\theta} \quad . \quad . \quad . \quad . \quad (2)$$

But the equation to the ellipse referred to its focus is

$$\varrho = \frac{b^2}{a} \times \frac{1}{1+e\cos.\theta}$$

$$\therefore \frac{b^2}{a} = \frac{2P(D-P)}{D}$$

and

$$e^2 = \frac{a^2-b^2}{a^2} = 1-\frac{b^2}{a^2} = \left(1-\frac{2P}{D}\right)^2$$

$$\therefore \frac{b^2}{a^2} = \frac{4P}{D} - \frac{4P^2}{D^2} = \frac{4P}{D^2} \times (D - P)$$

$$= \frac{b^2}{a} \times \frac{2}{D}$$

$$\left.\begin{array}{l} \therefore a = \dfrac{D}{2} \\ \text{and} \\ b = \sqrt{P \times (D - P)} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (3)$$

To find the Periodic Time; let $\theta = \pi$. Then $\varrho = 2a - P = D - P$, and equation (1) gives

$$\frac{T}{2} = \sqrt{\frac{D}{2g\mu}} \times \frac{D}{2} \times \left(\sin.^{-1} - 1 - \frac{\pi}{2}\right)$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{g\mu}} \times \left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$$

$$\therefore T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{g\mu}},$$

see 159.

OTHERWISE.

First find the Trajectory by formula (11. 307); then substitute for $\frac{\varrho^2}{p^2}$ in 9 and 10, &c.

310. *Required the Time and Trajectory when* $F = \frac{\mu}{\varrho^3}$.

By 304,

$$v^2 = -g\mu \times (D^{-2} - \varrho^{-2})$$

$$= \frac{g\mu}{D^2} \times \frac{D^2 - \varrho^2}{\varrho^2}$$

$\therefore$ if V and P belong to an apse we have

$$V^2 = \frac{g\mu}{D^2} \times \frac{D^2 - P^2}{P^2}$$

$$\therefore \varrho^2 v - P^2 V^2 = \frac{g\mu}{D^2} \times \{D^2 - \varrho^2 - D^2 + P^2\}$$

$$= \frac{g\mu}{D^2}(P^2 - \varrho^2)$$

$$\therefore t = \frac{D}{\sqrt{g\mu}} \int \frac{\varrho\, d\varrho}{\sqrt{P^2 - \varrho^2}}$$

$$= \frac{D}{\sqrt{g\mu}} \times (C \pm \sqrt{P^2 - \varrho^2})$$

and taking $t = 0$ at an apse or when $\varrho = P$, $C = 0$,

$$t = \frac{D}{\sqrt{g\mu}} \times \sqrt{P^2 - \varrho^2} \quad \ldots \ldots \ldots \quad (1)$$

also

$$\frac{\theta}{PV} = \int \frac{dt}{\varrho^2} = \frac{D}{\sqrt{g\mu}} \times \int \frac{d\varrho}{\varrho\sqrt{(P^2 - \varrho^2)}}$$

But

$$\int \frac{d\varrho}{\varrho\sqrt{(P^2 - \varrho^2)}} = \frac{1}{\pm P} \times \left\{ l.\frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho} + C \right\}$$

and

$$V = \frac{\sqrt{g\mu}}{PD} \times \sqrt{(D^2 - P^2)}.$$

$$\therefore \frac{\pm P\theta}{\sqrt{(D^2 - P^2)}} = l.\frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho} + C$$

and making $\theta = 0$ at the apse or where $\varrho = P$,

$$C = -l.\frac{P}{P} = 0$$

$$\therefore \pm \frac{P\theta}{\sqrt{D^2 - P^2}} = l.\frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho}$$

$$\therefore \pm \frac{P\theta}{e\sqrt{(D^2 - P^2)}} = \frac{\sqrt{(P^2 - \varrho^2)} \mp P}{\varrho}$$

which gives

$$\varrho = \frac{2Pe^{\frac{P\theta}{\sqrt{(D^2 - P^2)}}}}{1 + e^{\frac{2P\theta}{\sqrt{(D^2 - P^2)}}}} \quad \ldots \ldots \ldots \quad (2)$$

311. *Required the Trajectory and circumstances of motion when*

$$F = \frac{\mu}{\varrho^n}$$

or for any inverse law of the distance.

The readiest method is this; By (11) 307, if r, and P be the values of ϱ and p for the given velocity V (P is no longer an apsidal distance)

$$\frac{P^2V^2}{p^2} = V^2 + \frac{2\mu}{(n-1)r^{n-1}} \times \frac{r^{n-1} - \varrho^{n-1}}{\varrho^{n-1}} \quad \ldots \quad (1)$$

the equation to the Trajectory.

Also since

$$v\,dv = -gF\,d\varrho$$

$$\therefore v^2 = \frac{2 g \mu}{(n-1) \varrho^{n-1}} \text{ (from } \infty \text{ to } \varrho)$$

Hence

$$V^2 = \frac{2 g \mu}{(n-1) r^{n-1}} \text{ (from } \infty \text{ to } \varrho)$$

and if we put

$$\dot{V}^2 = \frac{2 m g \mu}{(n-1) r^{n-1}}$$

in which m may be $>$ $=$ or $<$ 1 we easily get

$$\left.\begin{aligned}
p &= \sqrt{\frac{m}{m-1}} \times \frac{P \varrho^{\frac{n-1}{2}}}{\sqrt{\left(\varrho^{n-1} + \frac{r^{n-1}}{m-1}\right)}} \; .. m > 1 \\
p &= \frac{P}{r^{\frac{n-1}{2}}} \times \varrho^{\frac{n-1}{2}} \; \ldots\ldots\ldots \; m = 1 \\
p &= \frac{\sqrt{\frac{m}{m-1}} \times P \varrho^{\frac{n-1}{2}}}{\sqrt{\left(\frac{r^{n-1}}{1-m} - \varrho^{n-1}\right)}} \; \ldots \; m < 1
\end{aligned}\right\} \; \ldots \; (2)$$

To find θ on this hypothesis.

We have (307)

$$d\theta = \frac{r p d\varrho}{\varrho \sqrt{(\varrho^2 - p^2)}}$$

which gives by substitution

$$d\theta = \pm r \sqrt{\frac{m}{m-1}} P \times \frac{\varrho^{\frac{n-3}{2}} d\varrho}{\varrho \sqrt{\left(\varrho^{n-1} - \frac{m}{m-1} P^2 \varrho^{\frac{n-3}{2}} + \frac{r^{n-1}}{m-1}\right)}} \; .. m > 1$$

$$d\theta = \pm \frac{r \varrho^{\frac{n-5}{2}} d\varrho}{\sqrt{\left(\frac{r^{n-1}}{P^2} - \varrho^{n-3}\right)}} \; \ldots\ldots\ldots\ldots \; m = 1$$

$$d\theta = \pm r \sqrt{\frac{m}{1-m}} \times P \times \frac{\varrho^{\frac{n-3}{2}} d\varrho}{\varrho \sqrt{\left(\frac{r^{n-1}}{1-m} - \frac{m}{1-m} P^2 \varrho^{n-3} - \varrho^{n-1}\right)}}$$

the positive or negative sign being used according as the body ascends or descends.

Ex. If n = 2, we get

$$p = \sqrt{\frac{m}{m-1}} P \times \frac{\varrho}{\sqrt{\left(\varrho^2 + \frac{r}{m-1} \varrho\right)}} \; \ldots \; m > 1$$

$$p = \frac{P}{r^{\frac{1}{2}}} \cdot \varrho^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots \quad m = 1$$

$$p = \sqrt{\frac{m}{1-m}}\, P \cdot \frac{\varrho}{\sqrt{\left(\frac{r\varrho}{1-m} - \varrho^2\right)}} \quad \ldots \quad m < 1$$

the equations to the ellipse, parabola and hyperbola respectively.

Also we have correspondingly

$$d\theta = \pm \frac{r\sqrt{\frac{m}{m-1}}\, P\, d\varrho}{\varrho\sqrt{\left(\varrho^2 + \frac{r}{m-1}\varrho - \frac{m}{m-1} \cdot P^2\right)}}$$

$$d\theta = \pm r\, P \cdot \frac{d\varrho}{\varrho\sqrt{(r\varrho - P^2)}}$$

$$d\theta = \pm r\sqrt{\frac{m}{1-m}} \cdot \frac{d\varrho}{\varrho\sqrt{\left(-\frac{m}{1-m}P^2 + \frac{r}{1-m}\varrho - \varrho^2\right)}}$$

which are easily integrated.

Ex. 2. Let $n = 3$. Then we get

$$p = \sqrt{\frac{m}{m-1}} \times P \times \frac{\varrho}{\sqrt{\left(\varrho^2 + \frac{r^2}{m-1}\right)}} \quad \ldots \quad m > 1$$

$$p = \frac{P}{r}\varrho \quad \ldots\ldots\ldots\ldots \quad m = 1$$

$$p = \sqrt{\frac{m}{m-1}} \times P \times \frac{\varrho}{\sqrt{\left(\frac{r^2}{1-m} - \varrho^2\right)}} \quad \ldots \quad m < 1$$

$$d\theta = \pm\sqrt{\frac{m}{m-1}} \cdot P r \times \frac{d\varrho}{\varrho\sqrt{\left(\varrho^2 - \frac{m P^2 - r^2}{m-1}\right)}} \quad . \ m > 1$$

$$d\theta = \pm \frac{P r}{\sqrt{(r^2 - P^2)}} \cdot \frac{d\varrho}{\varrho} \quad \ldots\ldots\ldots \quad m = 1$$

$$d\theta = \pm\sqrt{\frac{m}{1-m}} \times r P \times \frac{d\varrho}{\varrho\sqrt{\left(\frac{r^2 - m P^2}{1-m} - \varrho^2\right)}} \quad . \ m < 1$$

312. In the first of these values of θ, $m P^2$ may be $>$ $=$ or $<$ r^2.

(1). Let $m P^2 > r^2$. Then (see 86)

$$\theta - \alpha = \pm\sqrt{\frac{m}{m P^2 - r^2}} \times P r \left(\sec.^{-1} \varrho\sqrt{\frac{m-1}{m P^2 - r^2}} - \sec.^{-1} r\sqrt{\frac{m-1}{m P^2 - r^2}}\right)$$

and at an apse or when $r = P$

$$\theta = \pm\sqrt{\frac{m}{m-1}} \times P \times \sec.^{-1}\frac{\varrho}{P} \quad \ldots \quad (a)$$

for

$$\sqrt{\frac{m-1}{m P^2 - r^2}} = \frac{1}{P} \text{ or } = \frac{1}{r}.$$

(2) Let $m P^2 = r^2$. Then we have

$$p = \frac{\frac{r}{\sqrt{(m-1)}}\varrho}{\sqrt{\left(\varrho^2 + \frac{r^2}{m-1}\right)}} \quad \ldots\ldots\ldots\ldots \quad \text{(b)}$$

$$\theta - \alpha = \pm \frac{r^2}{\sqrt{(m-1)}} \times \int \frac{d\varrho}{\varrho^2}$$

$$= \pm \frac{r^2}{\sqrt{m-1}} \times \left(\frac{1}{r} - \frac{1}{\varrho}\right)$$

$$= \pm \frac{r}{\sqrt{m-1}} \times \frac{\varrho - r}{\varrho} \quad \ldots\ldots\ldots \quad \text{(c)}$$

which indicates the *Reciprocal* or *Hyperbolic Spiral.*

(3) Let $m P^2$ be $< r^2$. Then

$$p = \frac{\sqrt{\frac{m}{1-m}}\, P\varrho}{\sqrt{\left(\frac{r^2}{1-m} + \varrho^2\right)}} \quad \ldots\ldots\ldots\ldots \quad \text{(d)}$$

$$\theta = \pm r P \sqrt{\frac{m}{m-1}} \int \frac{d\varrho}{\varrho\sqrt{\left(\varrho^2 + \frac{r^2 - m P^2}{m-1}\right)}}$$

$$= \pm r P \sqrt{\frac{m}{r^2 - mP^2}} \times 1.\frac{\sqrt{\left(\varrho^2 + \frac{r^2 - mP^2}{m-1}\right)} - \sqrt{\frac{r^2 - mP^2}{m-1}}}{\varrho} + C$$

$$= \pm r P \sqrt{\frac{m}{r^2 - mP^2}} \times 1.\frac{r}{\varrho}\,\frac{\sqrt{(\overline{m-1}.\varrho^2 + r^2 - mP^2)} - \sqrt{(r^2 - mP^2)}}{\sqrt{m}.(r^2 - P^2) - \sqrt{(r^2 - m P^2)}} \quad \ldots \quad \text{(e)}$$

at an apse $r = P$; and then

$$\theta = \pm P \sqrt{\frac{m}{1-m}} \times 1.\frac{\sqrt{(r^2 - \varrho^2)} - r}{\varrho} \quad \ldots \quad \text{(f)}$$

Thus the first of the values of θ has been split into three, and integrating the other two we also get

$$\theta - \alpha = \pm \frac{P r}{\sqrt{(r^2 - P^2)}} \times (1\,\varrho - 1\,r)$$

$$= \pm \frac{P r}{\sqrt{(r^2 - P^2)}} \times 1.\frac{\varrho}{r}$$

$$\theta - \alpha = \pm r P \sqrt{\frac{m}{1-m}} \int \frac{d\varrho}{\varrho\sqrt{\left(\frac{r^2 - m P^2}{1-m} - \varrho^2\right)}}$$

$$= \pm r P \sqrt{\frac{m}{r^2 - mP^2}} \times l. \frac{\sqrt{\left(\frac{r^2 - mP^2}{1 - m} - \rho^2\right)} \sqrt{\frac{r^2 - mP^2}{1 - m}}}{\rho} + C$$

$$= \pm r P \sqrt{\frac{m}{r^2 - mP^2}} \times l. \frac{r}{\rho} \frac{\sqrt{(r^2 - mP^2 - \overline{1 - m}.\rho^2)} - \sqrt{(r^2 - mP^2)}}{\sqrt{(m.\overline{r^2 - \rho^2})} - \sqrt{(r^2 - mP^2)}}$$

and if θ is measured from an apse or $r = P$ it reduces to

$$\theta = \pm P \sqrt{\frac{m}{1 - m}} l. \frac{r + \sqrt{\rho^2 - r^2}}{\rho}.$$

313. Hence recapitulating we have these pairs of equations, viz.

$$(1)\ p = \frac{P\sqrt{\frac{m}{m-1}}\,\rho}{\sqrt{\left(\rho^2 + \frac{r^2}{m-1}\right)}}$$

$$\theta - \alpha = \pm P r \sqrt{\frac{m}{m P^2 - r^2}} \times \left(\sec.^{-1} \rho \sqrt{\frac{m-1}{m P^2 - r^2}} - \sec.^{-1} \sqrt{\frac{m-1}{m P^2 - r^2}}\right)$$

or

$$\theta = \pm P \sqrt{\frac{m}{m-1}} \times \sec.^{-1} \frac{\rho}{P}.$$

To construct the Trajectory,

put $\theta = 0$, then

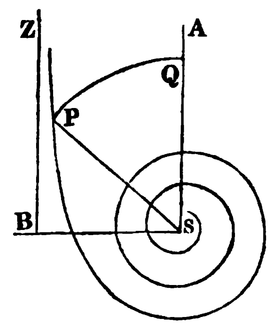
$\rho = P = SA$;

let $\rho = \infty$, then

$$p = \pm P \sqrt{\frac{m}{m-1}}$$

and

$$\theta = \pm \frac{P \pi}{2} \sqrt{\frac{m}{m-1}};$$



and taking A S B, A S B′ for these values of θ, and S B, S B′ for those of p and drawing B Z, B′ Z′ at right angles we have two asymptotes; S C being found by putting $\theta = \pi$. Thus and by the rules in (35, 36, 37, 38.) the curve may be traced in all its ramifications.

$$2.\ p = \frac{r}{\sqrt{(m-1)}} \times \frac{\rho}{\sqrt{\left(\rho^2 + \frac{r^2}{m-1}\right)}}$$

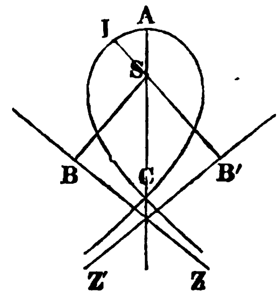
and

$$\theta - \alpha = \pm \frac{r}{\sqrt{(m-1)}} \times \frac{\rho - r}{\rho}$$

This equation becomes more simple when we make θ originate from $\varrho = \infty$; for then it is

$$\theta = \frac{r^2}{\sqrt{(m-1)}} \times \frac{1}{\varrho}$$

and following the above hinted method the curve, viz. the *Reciprocal Spiral*, may easily be described as in the annexed diagram.



3. $p = P\sqrt{\frac{m}{1-m}} \times \frac{\varrho}{\sqrt{\left(\frac{r^2}{1-m} + \varrho^2\right)}}$

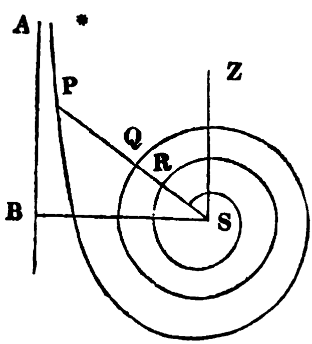
and

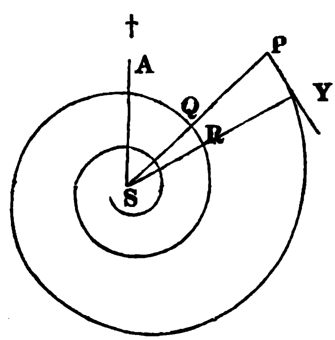
$$\theta - \alpha = \pm r P \sqrt{\frac{m}{r^2 - m P^2}} \times l.\frac{r}{\varrho}.\frac{\sqrt{(\overline{m-1}\,\varrho^2 + r^2 - mP^2)} - \sqrt{(r^2 - mP^2}}{\sqrt{m(r^2 - P^2)} - \sqrt{(r^2 - mP^2)}}$$

and when θ is measured from an apse or when $P' = r$

$$\theta = \pm P.\sqrt{\frac{m}{1-m}}.\,l\,\frac{\sqrt{(r^2 + \varrho^2)} - r}{\varrho}$$

Whence may easily be traced this figure.*





4. $p = \frac{P}{r}\varrho$

and

$$\theta - \alpha = \pm \frac{P r}{\sqrt{(r^2 - P^2)}}\, l.\frac{\varrho}{r}.$$

From which may be described the *Logarithmic Spiral*.†

5. $p = P\sqrt{\frac{m}{m-1}} \times \frac{\varrho}{\sqrt{\left(\frac{r^2}{1-m} - \varrho^2\right)}}$

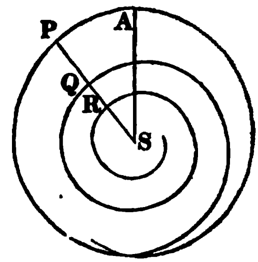
$$\theta - \alpha = \pm r P\sqrt{\frac{m}{r^2 - m P^2}} \times l.\frac{r}{\varrho}.\frac{\sqrt{(r^2 - m P^2 - \overline{1-m}.\varrho^2)} - \sqrt{(r^2 - m P^2)}}{\sqrt{(m.\overline{r^2 - \varrho^2})} - \sqrt{(r^2 - m P^2)}}$$

or

$$\theta = \pm r \sqrt{\frac{m}{1-m}}\, l.\, \frac{r - \sqrt{(\rho^2 - r^2)}}{\rho}$$

when $P = r$.

Whence this spiral.



These several spirals are called *Cotes' Spirals*, because he was the first to construct them as Trajectories.

314. If $n = 4$. Then the Trajectory, &c. are had by the following equations, viz.

$$p = \frac{P\sqrt{\frac{m}{m-1}} \cdot \rho^{\frac{3}{2}}}{\sqrt{\left(\rho^3 + \frac{r^3}{m-1}\right)}}$$

$$d\theta = r P \sqrt{\frac{m}{m-1}} \times \frac{d\rho}{\rho^{\frac{1}{2}}\sqrt{\left(\rho^3 - \frac{m}{m-1} P^2 \rho + \frac{r^3}{m-1}\right)}}$$

315. If $n = 5$. Then

$$p = P\sqrt{m}\, \frac{\rho^2}{\sqrt{(m-1 \cdot \rho^4 + r^4)}}$$

$$d\theta = r P \sqrt{\frac{m}{m-1}} \times \frac{d\rho}{\sqrt{\left(\rho^4 - \frac{m}{m-1} P^2 \rho^2 + \frac{r^4}{m-1}\right)}}$$

which as well as the former expression is not integrable by the usual methods.

When

$$\rho^4 - \frac{m}{m-1} P^2 \rho^2 + \frac{r^4}{m-1}$$

is a perfect square, or when

$$\frac{r^4}{m-1} = \frac{m^2 P^4}{4(m-1)^2}$$

then we have

$$d\theta = \pm r P \sqrt{\frac{m}{m-1}} \times \frac{d\rho}{\rho^2 - \cdot \frac{m P^2}{2(m-1)}}$$

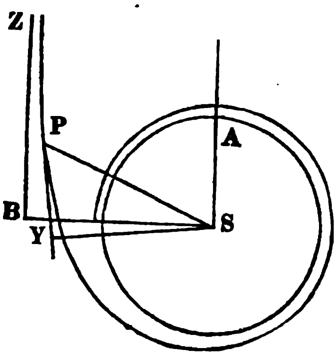
Therefore (87)

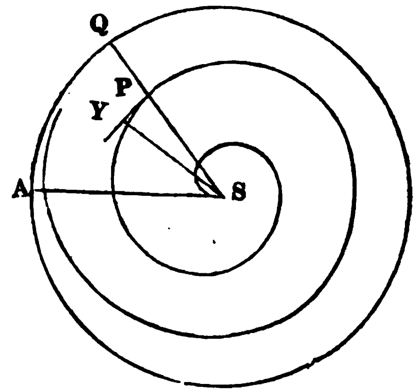
$$\theta - \alpha = r P \sqrt{\frac{m}{m-1}} \times \sqrt{\frac{2(m-1)}{m P^2}} \times l.\, \frac{\sqrt{\frac{m P^2}{2(m-1)}} - \rho}{\sqrt{\left(\rho^2 - \frac{m P^2}{2(m-1)}\right)}} + C$$

$$\theta - \alpha = r \sqrt{2} \times l. \frac{P\sqrt{m} - \varrho\sqrt{2(m-1)}}{\sqrt{(2.\overline{m-1}.\varrho^2 - m P^2)}}$$

$$\alpha - \theta \text{ or} = r \sqrt{2}\, l. \frac{\varrho\sqrt{2(m-1)} + P\sqrt{m}}{\sqrt{(m P^2 - 2.\overline{m-1}.\varrho^2)}}$$

and these being constructed will be as subjoined.





316. Cor. 1. otherwise.

To find the apses of an orbit where $F = \frac{\mu}{\varrho^n}$.

Let

$$p = \varrho.$$

Then

$$\left.\begin{array}{ll} \varrho^{n-1} - \frac{m}{m-1} P^2 \varrho^{n-3} + \frac{r^{n-1}}{m-1} = 0 & m > 1 \\ \varrho = \frac{r^{\frac{n-1}{n-3}}}{P^{\frac{2}{n-3}}} \ \ldots\ldots\ldots\ldots & m = 1 \\ \text{and} & \\ \varrho^{n-1} + \frac{m}{1-m} P^2 \varrho^{n-3} - \frac{r^{n-1}}{1-m} \ldots & m < 1 \end{array}\right\} \ldots A$$

which being resolved all the possible values of ϱ will be discovered in each case, and thence by substituting in θ, we get the position as well as the number of apses.

Ex. 1. Let n = 2. Then

$$\varrho^2 + \frac{r}{m-1}.\varrho - \frac{m P^2}{m-1} = 0$$

$$\varrho = \frac{P^2}{r} = \frac{\frac{L}{4} r}{r} = \frac{L}{4}$$

$$\varrho^2 - \frac{r}{1-m}\varrho + \frac{m P^2}{1-m} = 0$$

which give

$$\varrho = -\frac{r}{2(m-1)} \pm \sqrt{\frac{r^2 + 4mP^2.(m-1)}{4(m-1)^2}}$$

$$\varrho = \frac{L}{4}$$

and

$$\varrho = \frac{r}{2(1-m)} \pm \sqrt{\frac{r^2 - 4mP^2.(1-m)}{4.(1-m)^2}}.$$

Whence in the ellipse and hyperbola there are two apses (force in the focus); in the former lying on different sides of the focus; in the latter both lying on the same side of the focus, as is seen by substituting the values of ϱ in those of θ. Also there is but one in the parabola.

Ex. 2. Let n = 3. Then eq. (A) become

(1) $$\varrho^2 = \frac{mP^2 + r^2}{m-1}$$

which indicate two apses in the same straight line, and on different sides of the center, whose position will be given by hence finding θ;

(2) $$\varrho = \frac{r^{\frac{2}{0}}}{P^{\frac{2}{0}}} = \infty$$

because r is > P,

whence there is no apse.

(3) $$\varrho^2 = \frac{r^2 - mP^2}{1-m}$$

which gives two apses, r^2 being > mP^2 because m is < 1 and P < r; and their position is found from θ.

317. COR. 2. This is done also by the equation

$$p = \varrho.\sin.\varphi, \text{ or } \sin.\varphi = \frac{p}{\varrho}$$

φ being the ∠ required.

Ex. When n = 3, and m = 1, we have (4. 313)

$$p = \frac{P}{r}\varrho$$

$$\therefore \sin.\varphi = \frac{P}{V}$$

$\therefore \varphi$ is constant, a known property of the logarithmic spiral.

318. *To find when the body reaches the center of force.*

Put in the equations to the Trajectory involving p, ϱ; or ϱ, θ

$$\varrho = 0.$$

Ex. 1. When n = 3, in all the five cases it is found that

$$p = 0$$

and

$$\theta = -\infty.$$

Ex. 2. When $n = 5$ in the particular case of 315, the former value of θ becomes impossible, being the logarithm of a negative quantity, and the latter is $= -\infty$.

319. *To find when the Trajectory has an asymptotic circle.*

If at an apse for $\theta = \infty$ the velocity be the same as that in a circle at the same distance (166), or if when

$$\theta = \infty$$

and

$$p = \varrho$$

we also have

$$\frac{p}{\varrho} = \frac{d\,p}{d\,\varrho}$$

then it is clear there is an asymptotic circle.

Examples are in hypothesis of 315.

320. *To find the number of revolutions from an apse to* $\varrho = \infty$.

Let θ' be the value of $\theta - \alpha$ when $\varrho = p$ or at an apse, and θ'' when $\varrho = \infty$. Then

$$v = \frac{\theta' \sim \theta''}{2\,\pi} = \text{the number of revolutions required.}$$

Ex. By 313, we have

$$\theta' \sim \theta'' = P\sqrt{\frac{m}{m-1}}\,\sec.^{-1}\frac{\infty}{P}$$

$$= \sqrt{\frac{m}{m-1}}\cdot\frac{\pi}{2}$$

$$\therefore v = \tfrac{1}{4}\sqrt{\frac{m}{m-1}}.$$

321. Cor. 3. First let V R S be an hyperbola whose equation, x being measured from C, is

$$y^2 = \frac{b^2}{a^2}\cdot(x^2 - a^2).$$

Then

$$V\,C\,R = \frac{y \times x}{2} - \int y\,d\,x$$

But

$$\int y\,d\,x = \frac{b}{a}\int d\,x\,\sqrt{(x^2 - a^2)}$$

$$= \frac{b}{a}\,x\,\sqrt{x^2 - a^2} - \frac{b}{a}\int\frac{x^2\,d\,x}{\sqrt{(x^2 - a^2)}}$$

$$=\frac{b}{a}x\sqrt{(x^2-a^2)}-\frac{b}{a}\int dx\sqrt{(x^2-a^2)}-\frac{b}{a}\int\frac{a^2\,dx}{\sqrt{(x^2-a^2)}}$$

$$\therefore 2\int y\,dx=\frac{b}{a}x\sqrt{(x^2-a^2)}-a\,b\,l.\frac{x+\sqrt{(x^2-a^2)}}{a}$$

and

$$V\,C\,R=\frac{a\,b}{2}\,l.\frac{x+\sqrt{(x^2-a^2)}}{a}\quad\ldots\ldots\quad(1)$$

Again

$$\varrho=C\,P=C\,T=x-\text{subtangent}$$

$$=x-\frac{y\,dx}{dy}\ (29)$$

$$=x-\frac{x^2-a^2}{x}=\frac{a^2}{x}$$

and substituting for x in (1) we have

$$V\,C\,R=\frac{a\,b}{2}.\,l.\,\frac{\frac{a^2}{\varrho}+\sqrt{\left(\frac{a^4}{\varrho^2}-a^2\right)}}{a}$$

$$\therefore\theta=V\,C\,P\propto V\,C\,R=\frac{a\,b}{2}N\,l.\frac{a+\sqrt{(a^2-\varrho^2)}}{a\,\varrho}\quad\ldots\ldots\quad(2)$$

N being a constant quantity.

322. Hence conversely

$$\frac{2}{C^{abN}}\theta=\frac{1}{\varrho}-\sqrt{\left(\frac{1}{\varrho^2}-\frac{1}{a^2}\right)}$$

$$=u-\sqrt{\left(u^2-\frac{1}{a^2}\right)}$$

and differentiating (17) we get

$$\frac{d\,u^2}{d\,\theta^2}=\frac{4}{a^2\,b^2\,N^2}\times\left(u^2-\frac{1}{a^2}\right)\quad\ldots\ldots\quad(1)$$

and again differentiating ($d\,\theta$ being constant)

$$\frac{d^2u}{d\,\theta^2}=\frac{4}{a^2\,b^2\,N^2}\times U$$

Hence (139)

$$F=\frac{P^2\,V^2}{g}.\left(\frac{4}{a^2\,b^2\,N^2}+1\right)\frac{1}{\varrho^3}\propto\frac{1}{\varrho^3}$$

322. By the text it would appear that the body must proceed from V in a direction perpendicular to C V — i. e. that V is an apse.

From (1) 322, we easily get

$$\frac{d\,\varrho^2}{d\,\theta^2}=\frac{4}{a^4\,b^2\,N^2}(a^2\,\varrho^2-\varrho^4)$$

and since generally

$$p^2 = \frac{\varrho^4 d\theta^2}{\varrho^2 d\theta^2 + d\varrho^2}$$

$$\frac{d\varrho^2}{d\theta^2} = \frac{\varrho^2}{p^2} \cdot (\varrho^2 - p^2)$$

$$\therefore \frac{4}{a^4 b^2 N^2} \times p^2 \times (a^2 - \varrho^2) = \varrho^2 - p^2$$

$$\therefore p^2 = \frac{\varrho^2}{1 + \frac{4}{a^4 b^2 N^2} \times (a^2 - \varrho^2)} \quad . \quad . \quad . \quad . \quad (1)$$

which is another equation to the trajectory involving the perpendicular upon the tangent.

Now at an apse

$$p = \varrho$$

and substituting in equation (1) we get easily

$$\varrho = a$$

which shows V to be an apse.

OTHERWISE.

Put $d\varrho = 0$, for ϱ is then = max. or min.

324. *With a proper velocity.*]

The velocity with which the body must be projected from V is found from

$$v\,dv = -g\,F\,d\varrho.$$

325. *Descend to the center*]. When

$$\varrho = 0,\ p = 0\ (1.\ 323) \text{ and } \theta = \infty\ \ (2.\ 321).$$

326. Secondly, let V R S be an ellipse, whose equation referred to the center C is

$$y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2)$$

Then

$$V\,C\,R = \frac{y\,x}{2} + \int - y\,dx$$

$$= \frac{y\,x}{2} - \frac{b}{a} \int dx\,\sqrt{(a^2 - x^2)}$$

and as above, integrating by parts,

$$\int dx\,\sqrt{(a^2 - x^2)} = \frac{x\,\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2} \int \frac{dx}{\sqrt{(a^2 - x^2)}}$$

$$= \frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\left(\sin.^{-1}\frac{x}{a} - \frac{\pi}{2}\right)$$

$$\therefore V C R = \frac{a b}{2}\left(\frac{\pi}{2} - \sin.^{-1}\frac{x}{a}\right).$$

Also

$$\varrho = x + N T = x + \frac{-y d x}{d y}$$

$$= x + \frac{a^2 - x^2}{x} = \frac{a^2}{x}$$

and

$$\theta = N . V C R = \frac{a b N}{2} . \left(\frac{\pi}{2} - \sin.^{-1}\frac{a}{\varrho}\right) \ldots (1)$$

$$\therefore \sin.^{-1}\frac{a}{\varrho} = \frac{\pi}{2} - \frac{2\theta}{a b N}$$

$$\therefore \frac{a}{\varrho} = \sin.\left(\frac{\pi}{2} - \frac{2\theta}{a b N}\right) = \cos.\frac{2\theta}{a b N}$$

and

$$\varrho = a \sec.\frac{2\theta}{a b N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Conversely by the expression for F in 139, we have

$$F \propto \frac{1}{\varrho^3}$$

327. *To find when the body is at an apse, either proceed as in 323, or put*

$$d\varrho = 0.$$

By (27) $d . \sec. x = \dfrac{d x . \sin. x}{\cos.^2 x}$

$$\therefore \frac{\sin.\theta}{\cos.^2\theta} = 0$$

or

$$\theta = 0$$

that is the point V is an apse.

328. The proper velocity of projection is easily found as indicated in 324.

329. *Will ascend perpetually and go off to infinity.*]

From (2) 327, we learn that when

$$\frac{2\theta}{a b N} = \frac{\pi}{2}$$

$$\varrho \text{ is } \infty;$$

also that ϱ can never $= 0$.

330. When the force is changed from centripetal to centrifugal, the sign of its expression (139) must be changed.

331. Prop. XLII. The preceding comments together with the Jesuits' notes will render this proposition easily intelligible.

The expression (139)

$$F = \frac{P^2 V^2}{g} \times \frac{d p}{p^3 d \varrho}$$

or rather (307)

$$p^2 = \frac{P^2 V^2}{V^2 - 2 g \int d \varrho f \varrho}$$

in which P and V are given, will lead to a more direct and convenient resolution of the problem.

It must, however, be remarked, that the difference between the first part of Prop. XLI. and this, is that the *force itself is given in the former and only the law of force* in the latter. That is, if for instance $F = \mu \varrho^{n-1}$, in the former μ is given, in the latter not. But since V is given in the latter, we have μ from 304.

SECTION IX.

332. Prop. XLIII. *To make a body move in an orbit revolving about the center of force, in the same way as in the same orbit quiescent*] that is, To adjust the angular velocity of the orbit, and centripetal force so that the body may be at any time at the same point in the revolving orbit as in the orbit at rest, and tend to the same center.

That it may tend to the same center (see Prop. II), the area of the new orbit in a fixed plane (V C p) must $\propto$ time $\propto$ area in the given orbit (V C P); and since these areas begin together their increments must also be proportional, that is (see fig. next Prop.)

$$CP \times KR \propto Cp \times kr$$

But

$$KR = CK \times \angle KCP$$

$$kr = Ck \times \angle kCp$$

and $CP = Cp$, and $CK = Ck$

$$\therefore \angle KCP \propto kCp$$

and the angles V C P, V Cp begin together

$$\therefore \angle VCP \propto \angle VCp.$$

Hence in order that the centripetal force in the new orbit may tend to C, it is *necessary* that

$$\angle \mathrm{VCp} \propto \angle \mathrm{VCP}.$$

Again, taking always

$$\mathrm{CP} = \mathrm{Cp}$$

and

$$\mathrm{VCp} : \mathrm{VCP} :: \mathrm{G} : \mathrm{F}$$

G : F being an invariable ratio, the equation to the locus of p or the orbit in fixed space can be determined; and thence (by 137, 139, or by Cor. 1, 2, 3 of Prop. VI) may be found the centripetal force in that locus.

333. *To find the orbit in fixed space or the locus of* p.

Let the equation to the given orbit V C P be

$$\varrho = \mathrm{f}(\theta)$$

where $\varrho = \mathrm{CP}$, and $\theta = \mathrm{VCP}$, and f means any function; then that of the locus is

$$\varrho = \mathrm{f}\left(\frac{\mathrm{G}}{\mathrm{F}}\theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which will give the orbit required.

OTHERWISE.

Let p′ be the perpendicular upon the tangent in the given orbit, and p that in the locus; then it is easily got by drawing the incremental figures and similar triangles (see fig. Prop. XLIV) that

$$\mathrm{KR} : \mathrm{kr} :: \mathrm{F} : \mathrm{G}$$
$$\mathrm{kr} : \mathrm{pr} :: \mathrm{p} : \sqrt{(\varrho^2 - \mathrm{p}^2)}$$
$$\mathrm{pr} : \mathrm{PR} :: 1 : 1$$
$$\mathrm{PR} : \mathrm{KR} :: \sqrt{(\varrho^2 - \mathrm{p}'^2)} : \mathrm{p}'$$

whence

$$1 : 1 :: \mathrm{F}.\mathrm{p}\sqrt{(\varrho^2 - \mathrm{p}'^2)} : \mathrm{Gp}'\sqrt{(\varrho^2 - \mathrm{p}^2)}$$

and

$$\therefore \mathrm{p}^2 = \frac{\mathrm{G}^2\mathrm{p}'^2\varrho^2}{\mathrm{F}^2\varrho^2 + (\mathrm{G}^2 - \mathrm{F}^2)\mathrm{p}'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

334. Ex. 1. Let the given Trajectory be the ellipse with the force in its focus; then

$$\mathrm{p}'^2 = \frac{\mathrm{b}^2\varrho}{2\mathrm{a} - \varrho}, \text{ and } \varrho = \frac{\mathrm{a}.(1 - \mathrm{e}^2)}{1 + \mathrm{e}\cos.\theta}$$

and therefore

$$\mathrm{p}^2 = \frac{\mathrm{b}^2\mathrm{G}^2(2\mathrm{a} - \varrho)\varrho^2}{\mathrm{b}^2(\mathrm{G}^2 - \mathrm{F}^2) + \mathrm{F}^2(2\mathrm{a}\varrho - \varrho^2)}$$

and

$$\varrho = \frac{a.(1 - e^2)}{1 + e \cos.\left(\frac{F}{G}\theta\right)}.$$

Hence since the force is (139)

$$\frac{P^2 V^2}{g} u^2\left(\frac{d^2 u}{d\theta^2} + u\right)$$

and here we have

$$a(1 - e^2) u = 1 + e \cos. \frac{F}{G}\theta$$

$$\therefore \frac{d u^2}{d\theta^2} = \frac{F^2}{a^2 G^2 (1 - c^2)^2} \times \{e^2 - (1 - a(1 - e^2) u)^2\}$$

$$= Q + \frac{2 F^2}{a G^2 (1 - e^2)} u - \frac{F^2}{G^2} u^2$$

and again differentiating, &c. we have

$$\frac{d^2 u}{d\theta^2} + u = \frac{F^2}{G^2 a(1 - e^2)} + \frac{G^2 - F^2}{G^2} \times u.$$

But if 2 R = latus-rectum we have

$$R = \frac{L}{2} = \frac{b^2}{a} = a(1 - e^2)$$

∴ the force in the new orbit is

$$\frac{P^2 V^2}{g R G^2} \times \left\{\frac{F^2}{\varrho^2} + \frac{R G^2 - R F^2}{\varrho^3}\right\}.$$

335. Ex. 2. Generally let the equations to the given trajectory be

$$\left.\begin{aligned} \varrho &= f(\theta') \\ \text{and} \quad p' &= \varphi(\varrho) \end{aligned}\right\}$$

Then since

$$\theta = \frac{G}{F}\theta'$$

$$\therefore d\theta^2 = \frac{G^2}{F^2} d\theta'^2$$

$$\therefore \frac{d^2 u}{d\theta^2} + u = \frac{F^2 d^2 u}{G^2 d\theta'^2} + u$$

$$= \frac{F^2}{G^2} \times \left(\frac{d^2 u}{d\theta^2} + u\right) + u - \frac{F^2}{G^2} u$$

and if the centripetal forces in the given trajectory and locus be named X, X′, by 139 we have

$$\frac{g X'}{P^2 V^2} = \frac{F^2}{G^2} \times \frac{g X}{P^2 V'^2} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^3}$$

$$\therefore X' = \frac{P^2 V^2}{G^2}\left(\frac{F^2 X}{P'^2 V'^2} + \frac{G^2 - F^2}{g} \times \frac{1}{\varrho^3}\right) \quad . \quad . \quad (1)$$

Also from (2. 333) we have

$$\frac{1}{p^2} = \frac{F^2}{G^2} \times \frac{1}{p'^2} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^2}$$

$$\therefore \frac{dp}{p^3 d\varrho} = \frac{F^2}{G^2} \times \frac{dp'}{p'^3 d\varrho} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^3}$$

$\therefore$ by 139

$$\frac{g X'}{P^2 V^2} = \frac{F^2 g X}{P'^2 V'^2} + \frac{G^2 - F^2}{G^2} \times \frac{1}{\varrho^3}$$

the same as before.

This second general example includes the first, as well as Prop. XLIV, &c. of the text.

336. *Another determination of the force tending to* C *and which shall make the body describe the locus of* p.

First, as before, we must show that in order to make the force X tend to C, the ratio $\angle$ V C P : $\angle$ V C p must be constant or = F : G.

Next, since they begin together the corresponding angular velocities ω, ω' of C P, C p are in that same ratio; i. e.

$$\omega : \omega' :: F : G.$$

Now in order to exactly counteract the centrifugal force which arises from the angular motion of the orbit, we must add the same quantity to the centripetal force. Hence if φ, φ' denote the centrifugal forces in the given orbit and the locus, we have

$$X' = X + \varphi' - \varphi$$

X being the force in the given orbit.

But (210)

$$\varphi = \frac{P^2 V^2}{g} \times \frac{1}{\varrho^3}$$

and

$$\propto \omega^2$$

when ϱ is given.

$$\therefore \varphi' = \varphi \times \frac{\omega'^2}{\omega^2} = \varphi \times \frac{G^2}{F^2} = \frac{P^2 V^2}{g} \times \frac{G^2}{F^2} \times \frac{1}{\varrho^3}$$

$$\therefore \varphi' - \varphi = \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

$$X' = X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$= \frac{P^2 V^2}{g} \times \left(u^2 . \overline{\frac{d^2 u}{d \theta^2} + u} + \frac{G^2 - F^2}{F^2 \rho^3}\right) \quad . \quad . \quad . \quad (2)$$

or

$$= \frac{P^2 V^2}{g} \times \left(\frac{d p}{p^3 d \rho} + \frac{G^2 - F^2}{F^2 \rho^3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

337. Prop. XLIV. Take u p, u k similar and equal to V P and V K; also

$$m r : k r :: \angle V C p : V C P.$$

Then since always C P = p c, we have

$$p r = P R.$$

Resolve the motions P K, p k into P R, R K and p r, r k. Then

$$R K (= r k) : r m :: \angle V C P : \angle V C p$$

and therefore when the centripetal forces P R, p r are equal, the body would be at m. But if

$$p C n : p C k :: V C p : V C P$$

and

$$C n = C k$$

the body will really be in n.

Hence the difference of the forces is

$$m n = \frac{m k \times m s}{m t} = \frac{(m r - k r) . (m r + k r)}{m t}.$$

But since the triangles p C k, p C n are given,

$$K r \propto m r \propto \frac{1}{C p}$$

$$\therefore m n \propto \frac{1}{C p^2} \times \frac{1}{m t}.$$

Again since

$$\begin{aligned} p C k : p C n &:: P C K : p C n :: V C P : V C p \\ &:: k r \quad : m r \quad \text{by construction} \\ &:: p C k : p C m \text{ ultimately} \end{aligned}$$

$$\therefore p C n = p C m$$

and m n ultimately passes through the center. Consequently

$$m t = 2 C p \text{ ultimately}$$

and

$$m n \propto \frac{1}{C p^3}$$

OTHERWISE.

338. By 336,

$$X' - X = \phi' - \phi$$
$$= \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$
$$\propto \frac{1}{\varrho^3}.$$

339. *To trace the variations of sign of* m n.

If the orbit move in consequentia, that is in the same direction as C P, the new centrifugal force would throw the body farther from the center, that is

C m is $>$ C n or C k

or m n is positive.

Again, when the orbit is projected in antecedentia with a velocity $<$ than twice that of C P, the velocity of C p is less than that of C P. Therefore

C m is $<$ C n

or m n is negative.

Again, when the orbit is projected in antecedentia with a velocity $=$ twice that of C P, the angular velocity of the orbit just counteracts the velocity of C P, and

$$m\,n = 0.$$

And finally, when the orbit is projected in antecedentia with a velocity $>$ 2 vel. of C P, the velocity of C p is $>$ vel. of C P or C m is $>$ C n, or

m n is positive.

OTHERWISE.

By 338,

$$m\,n \propto \phi' - \phi$$
$$\propto \omega'^2 - \omega^2$$

But

$$\omega' = \omega \pm W$$

W being the angular velocity of the orbit.

$$\therefore m\,n \propto \pm 2\,\omega W + W^2$$
$$\propto \pm 2\,\omega + W$$

$+$ or $-$ being used according as W is in consequentia or antecedentia.

Hence m n is positive or negative according as W is positive, and negative and $> 2\,\omega$; or negative and $< 2\,\omega$. That is, &c. &c.

Also when W is negative and $= 2\,\omega$, $m = 0$. Therefore, &c.

340. COR. 1. Let D be the difference of the forces in the orbit and in the locus, and f the force in the circle K R, we have

$$D : f :: m\,n : z\,r$$

$$:: \frac{m\,k \times m\,s}{m\,t} : \frac{r\,k^2}{2\,k\,c}$$

$$:: \frac{(m\,r + r\,k)\,(m\,r - r\,k)}{2\,k\,c} : \frac{r\,k^2}{2\,k\,c}$$

$$:: m\,r^2 - r\,k^2 : r\,k^2$$

$$:: G^2 - F^2 : F^2.$$

341. COR. 2. *In the ellipse with the force in the focus, we have*

$$X' \propto \frac{F^2}{A^2} + \frac{R\,G^2 - R\,F^2}{A^3}.$$

For (C V being put $=$ T)

$$\text{Force at V in Ellipse : Do. in circle} :: \frac{v^2}{\text{chord P V}} : \frac{v'^2}{P'\,V'}$$

$$:: \frac{1}{2\,R} : \frac{1}{2\,T}$$

$$:: T : R$$

$$\text{Also F in Circle : m n at V} :: F^2 : G^2 - F^2$$

$$\text{m n at V : m n at p} :: \frac{1}{T^3} : \frac{1}{A^3}$$

$$\therefore \text{ F at V in ellipse : m n at p} :: \frac{T\,F^2}{T^3} : \frac{R\,G^2 - R\,F^2}{A^3}.$$

Hence

$$X = \frac{F^2}{A^2}$$

we have

$$\text{F in ellipse at V} = \frac{F^2}{T^2}$$

and

$$m\,n = \frac{R\,G^2 - R\,F^2}{A^3}$$

and

$$X' = X + m\,n$$

$$\propto \frac{F^2}{A^2} + \frac{R\,G^2 - R\,F^2}{A^3}$$

see 334.

OTHERWISE.

342. By 336,

$$X' = X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

But

$$X = \frac{\mu}{\varrho^2}$$

and

$$\frac{P^2 V^2}{g} = \frac{L}{2} \mu = R \mu \text{ (157)}$$

$$\therefore X' = \frac{\mu}{F^2} \times \left\{ \frac{F^2}{\varrho^2} + \frac{G^2 - F^2}{\varrho^3} \right\}$$

343. COR. 3. *In the ellipse with the force in the center.*

$$X' \propto \frac{F^2 A}{T^3} + \frac{R G^2 - R F^2}{A^3}.$$

For here X $\propto$ A and the force generally $\propto \frac{v^2}{PV}$ (140)

$$\therefore \begin{cases} \text{Force in ellipse at V : Force in circle at V} :: T : R \\ \text{F in circle : m n at V} :: F^2 : G^2 - F^2 \\ \text{m n at V : m n at p} :: \frac{1}{T^3} : \frac{1}{A^3} \end{cases}$$

$$\therefore \text{F in ellipse at V : m n at p} :: \frac{F^2}{T^3} . T : \frac{R G^2 - R F^2}{A^3}$$

$\therefore$ assuming F in ellipse at P $= \frac{F^2 A}{T^3}$, we have

$$\text{F in ellipse at V} = \frac{F^2}{T^3} \times T$$

and

$$\therefore m n = \frac{R G^2 - R F^2}{A^3}$$

$$\therefore X' \propto X + m n \propto, \text{\&c.}$$

OTHERWISE.

344. $$X = \mu \varrho, \text{ and } \frac{P^2 V^2}{g} = \frac{4 \text{ (Area of Ellipse)}^2}{g \text{ (Period)}^2}$$

$$= \frac{4 \pi^2 a^2 b^2}{g \text{(Period)}^2} = \mu a^2 b^2 \text{ (147)}$$

Therefore by 336

$$X' = \mu \varrho + \mu a^2 b^2 \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

$$= \frac{\mu a^3}{F^2} \times \left\{ \frac{F^2 \varrho}{a^3} + \frac{b^2 \times (G^2 - F^2)}{a \varrho^3} \right\}$$

$$= \frac{\mu a^3}{F^2} \times \left\{ \frac{F^2 \varrho}{a^3} + \frac{R G^2 - R F^2}{\varrho^3} \right\}$$

345. Cor. 4. *Generally let* X *be the force at* P, $V \frac{F^2}{T^2}$ *at* V, R *the radius of curvature in* V, C V = T, &c. *then*

$$X' \propto X + \frac{V R G^2 - V R F^2}{A^3}$$

For

$$\left\{ \begin{array}{llll} \text{F in orbit at V} & : \text{F}' \text{ in circle at V} & :: T & : R \\ \text{F}' & : \text{m n at V} & :: F^2 & : G^2 - F^2 \\ \text{m n at V} & : \text{m n} & :: A^3 & : T^3 \end{array} \right.$$

$$\therefore \text{F in orbit at V} : \text{m n} :: \frac{V F^2}{T^2} : V R . \frac{G^2 - F^2}{A^3}$$

$\therefore$ since by the assumption

$$\text{F in orbit at V} = \frac{V F^2}{T^2}$$

$$\therefore \text{m n} = \frac{V R (G^2 - F^2)}{A^3}$$

and

$$X' \propto X + \frac{V R (G^2 - F^2)}{A^3}.$$

OTHERWISE.

This may better be done after 386, where it must be observed V is not the same as the indeterminate quantity V in this corollary.

346. Cor. 5. The equation to the new orbit is (333)

$$p^2 = \frac{G^2 p'^2 \varrho^2}{F^2 \varrho^2 + (G^2 - F^2) p'^2}$$

p′ belonging to the given orbit.

Ex. 1. *Let the given orbit be a common parabola.*

Then

$$p'^2 = r \varrho$$

$$\therefore p^2 = \frac{G^2 r \varrho^2}{F^2 \varrho + (G^2 - F^2) r}$$

and the new force is obtained from 336.

Ex. 2. *Let the given orbit be any one of Cotes' Spirals, whose general equation is*

$$p'^2 = \frac{b^2 \rho^2}{a^2 \mp \rho^2}.$$

Then the equation of 333 becomes

$$p^2 = \frac{\frac{G^2}{F^2} b^2 \rho^2}{\frac{G^2}{F^2} b^2 + a^2 - b^2 \mp \rho^2}$$

which being of the same form as the former shows the locus to be similar in each case to the given spiral.

This is also evident from the law of force being in each case the same (see 336) viz.

$$X' = \frac{\mu}{\rho^3} + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{G^2} \times \frac{1}{\rho^3}$$

$$\propto \frac{1}{\rho^3}.$$

Ex. 3. If the given orbit be a circle, the new one is also.

Ex. 4. *Let the given trajectory be a straight line.*

Here p' is constant. Therefore

$$p^2 = \frac{G^2}{F^2} \frac{p'^2 \times \rho^2}{\rho^2 + \frac{G^2 - F^2}{F^2} p'^2}$$

the equation to the elliptic spiral, &c. &c.

Ex. 5. *Let the given orbit be a circle with the force in its circumference.*

Here

$$p'^2 = \frac{\rho^2 (4 r^2 - \rho^2)}{4 r^2}$$

and we have from 333

$$p^2 = \frac{G^2 \rho^4}{4 r^2 F^2 + (G^2 - F^2) \rho^2}.$$

Ex. 6. *Let the given orbit be an ellipse with force in the focus.*

Here

$$p'^2 = \frac{b^2 \rho}{2 a - \rho}$$

and this gives

$$p^2 = \frac{G^2 b^2 \rho^2}{F^2 \rho (2 a - \rho) + b^2 (G^2 - F^2)}.$$

347. *To find the points of contrary flexure*, in the locus put

$$d p = 0;$$

and this gives in the case of the ellipse

$$\varrho = \frac{b^2}{a} \cdot \frac{F^2 - G^2}{F^2}$$

OTHERWISE.

In passing from convex to concave towards the center, the force in the locus must have changed signs. That is, at the point of contrary flexure, the force equals nothing or in this same case

$$F^2 A + R G^2 - R F^2 = 0$$

$$\therefore A = \frac{R}{F^2} \times (F^2 - G^2)$$

$$= \frac{b^2}{a} \cdot \frac{F^2 - G^2}{F^2}.$$

And generally by (336) we have in the case of a contrary flexure

$$X' = X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3} = 0$$

which will give all the points of that nature in the locus.

348. *To find the points where the locus and given Trajectory intersect one another.*

It is clear that at such points

$$\varrho = \varrho', \text{ and } \theta' = 2 W \pi \pm \theta$$

W being any integer whatever. But

$$\theta' = \frac{G}{F} \theta = m \theta$$

$$\therefore \theta = \frac{2 W \pi}{m \pm 1}.$$

This is independent of either the Trajectory or Locus.

349. *To find the number of such intersections during an entire revolution* of C P.

Since θ cannot be $> 2 \pi$

W is $< m + 1$ and also $< m - 1$

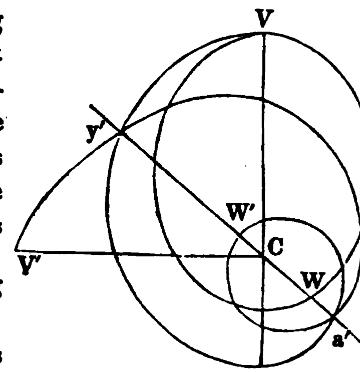
$\therefore 2 W$ is $< 2 m$.

Or the number required is the greatest integer in $2 m$ or $\frac{2 G}{F}$.

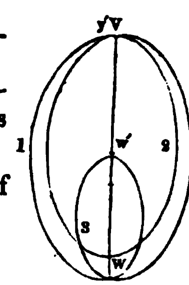
This is also independent of either Trajectory or Locus.

350. *To find the number and position of the double points of the Locus, i. e. of those points where it cuts or touches itself.*

Having obtained the equation to the Locus find its singular points whether double, triple, &c. by the usual methods; or more simply, consider the double points which are owing to apses and pairs of equal values of C P, one on one side of C V and the other on the other, thus:



The given Trajectory V W being symmetrical on either side of V W, let W′ be the point in the locus corresponding to W. Join C W′ and produce it indefinitely both ways. Then it is clear that W′ is an apse; also that the angle subtended by V v′ x′ W′ is $= \frac{G}{F} \times \pi = w\pi + \angle VCy'$, w being the greatest whole number in $\frac{G}{F}$ (this supposes the motion to be in consequentia). Hence it appears that whereever the Locus cuts the line C W′ there is a double point or an apse, and also that there are w + 1 such points.

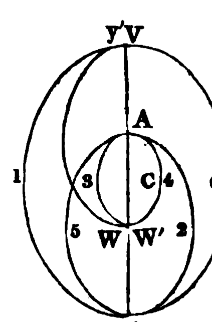


Ex. 1. Let $\frac{G}{F} = 2$; i. e. *let the orbit move in consequentia with a velocity = the velocity of* C P. Then $\angle VCy' = 0$, y′ coincides with V, and the double points are y′ V, x′ and W′.

The course of the Locus is indicated by the order of the figures 1, 2, 3, 4.

Ex. 2. Let $\frac{G}{F} = 3$.

Then the Locus resembles this figure, 1, 2, 3, 4, 5, 6. showing the course of the curve in which V, x′, A, W′ are double points and also apses.

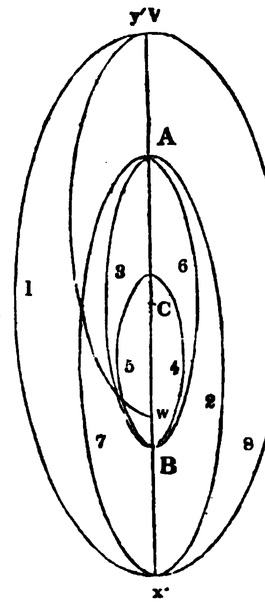


Ex. 3. Let $\frac{G}{F} = 4$.

Then this figure sufficiently traces the Locus. Its five double points, viz. V, x′, A, B, W′ are also apses.

Higher *integer* values of $\frac{G}{F}$ will give the Locus

still more complicated. If $\frac{G}{F}$ be not integer, the figure will be as in the first of this article, the double points lying out of the line C V. Moreover if $\frac{G}{F}$ be less than 1, or if the orbit move in antecedentia this method must be somewhat varied, but not greatly. These and other curiosities hence deducible, we leave to the student.



351. *To investigate the motion of* (p) *when the ellipse, the force being in the focus, moves in antecedentia with a velocity* = *velocity of* C P *in consequentia.*

Since in this case

$$G = 0$$

∴ (333) also

$$p = 0$$

or the Locus is the straight line C V.

Also (342)

$$X' = \frac{\mu}{F^2}\left(\frac{F^2}{\rho^2} - \frac{R\,F^2}{\rho^3}\right)$$

$$= \mu \times \frac{\rho - R}{\rho^2}.$$

Hence

$$v\,d\,v \propto X'\,d\,\rho \propto \frac{-d\,\rho}{\rho^2} + \frac{R\,d\,\rho}{\rho^3}$$

$$\therefore v^2 \propto \frac{2\,\rho - R}{\rho^2} - 1 \propto \frac{2\,\rho - \overline{1 - e^2} - \rho^2}{\rho^2}$$

(where $\frac{\text{axis major}}{2} = 1$;) and the body stops when

$$2\,\rho - 1 + e^2 - A^2 = 0,$$

or when

$$\rho = 1 \pm e.$$

Hence then the body moves in a straight line C V, the force increasing to $\frac{3}{4}$ of the latus-rectum from the center, when it = max. Then it decreases until the distance = $\frac{L}{2}$ or R. Here the centrifugal force prevails, but the velocity being then = max. the body goes forward till the

distance = the least distance when $v = 0$, and afterwards it is repelled and so on in infinitum.

352. *To find when the velocity in the Locus = max. or min.*

Since in either case

$$d \cdot v^2 = 2 v d v = 0$$

and

$$v d v = X' d \rho$$

$$\therefore X' = 0$$

$\therefore$ (336)

$$X + \frac{P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\rho^3} = 0$$

Ex. In the ellipse with the force in the focus, we have (342)

$$X' = \frac{\mu}{F^2}\left(\frac{F^2}{\rho^2} + \frac{R G^2 - R F^2}{\rho^3}\right)$$

$$\therefore \frac{F^2}{\rho^2} + \frac{R G^2 - R F^2}{\rho^3} = 0$$

$$\therefore \rho = R \times \frac{F^2 - G^2}{F^2}$$

$$= \frac{b^2}{a} \times \frac{F^2 - G^2}{F^2}.$$

If $G = 0$, $v = \max.$ when $\rho = \frac{b^2}{a} = \frac{L}{2}$, or when P is at the extremity of the latus-rectum.

If $F = 2 G$, $v = \max.$ when $\rho = R \cdot \frac{4 G^2 - G^2}{4 G^2} = \frac{3}{4} R = \frac{3}{8}$ lat. rectum.

353. *To find when the force X' in the Locus = max. or min.*

Put $d X' = 0$, which gives (see 336)

$$d X = \frac{3 P^2 V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\rho^4}$$

Ex. In the ellipse

$$X = \frac{\mu}{\rho^2}$$

and (157)

$$\frac{P^2 V^2}{g} = \mu R$$

$$\therefore \frac{-2 F^2 d \rho}{\rho^3} - \frac{3 R G^2 d \rho - 3 R F^2 d \rho}{\rho^4} = 0$$

which gives

$$\rho = \frac{3 R}{2} \times \frac{F^2 - G^2}{F^2}.$$

Hence when

$$G = 0$$

$$X = \text{max. when } \varrho = \frac{3\,R}{2}.$$

When $\varrho = R$, and $G = 0$. Then

$$X = \frac{F^2}{R^2} - \frac{R\,F^2}{R^3} = 0.$$

When $F = 2\,G$, or the ellipse moves in consequentia with $\frac{1}{2}$ the velocity of C p; then

$$X = \text{max. when}$$

$$\varrho = \frac{3\,R}{2} \cdot \frac{4\,G^2 - G^2}{4\,G^2} = \frac{9}{8}\,R.$$

354. Cor. 6. Since the given trajectory is a straight line and the *center of force* C *not in it*, this force cannot act at all upon the body, or (336)

$$X = 0.$$

Hence in this case

$$X' = \frac{P^2\,V^2}{g} \times \frac{G^2 - F^2}{F^2} \times \frac{1}{\varrho^3}$$

where $P = C\,V$ and V the given uniform velocity along V P.

In this case the Locus is found as in 346.

355. If the given Trajectory is a circle, it is clear that the Locus of p is likewise a circle, the radius-vector being in both cases invariable.

356. Prop. XLV. *The orbits (round the same center of force) acquire the same form, if the centripetal forces by which they are described at equal altitudes be rendered proportional.*]

Let f and f′ be two forces, then if at all equal altitudes

$$f \propto f'$$

the orbits are of the same form.

For (46)

$$f \propto \frac{d^2\varrho}{d\,t^2} \propto \frac{1}{d\,t^2} \propto \frac{1}{S\,P^2 \times Q\,T^2}$$

$$\propto \frac{1}{Q\,T^2} \propto \frac{1}{S\,P^2 \times d\,\theta^2}$$

$$\propto \frac{1}{d\,\theta^2}.$$

$$\therefore \frac{1}{d\,\theta^2} \propto \frac{1}{d\,\theta'^2}$$

and

$$d\,\theta \propto d\,\theta'.$$

But they begin together and therefore

$$\theta \propto \theta'$$

and

$$\varrho = \varrho'.$$

Hence it is clear the orbits have the same form, and hence is also suggested the necessity for making the angles θ, θ' proportional.

Hence then X′, and X being given, we can find $\frac{G}{F}$ such as shall render the Trajectory traced by p, very nearly a circle. This is done approximately by considering the given fixed orbit nearly a circle, and equating as in 336.

357. Ex. 1. *To find the angle between the apsides when* X′ *is constant.*

In this case (342)

$$X' \propto 1 \propto \frac{\varrho^3}{\varrho^3} \propto \frac{F^2 \varrho + R G^2 - R F^2}{\varrho^3}.$$

Now making $\varrho = T - x$, where x is indefinitely diminishable, and equating, we have

$$\begin{aligned}(T-x)^3 &= F^2 T - F^2 x + R G^2 - R F^2 \\ &= T^3 - 3 T^2 x + 3 T x^2 - x^3\end{aligned}$$

and equating homologous terms (6)

$$T^3 = F^2 T + R G^2 - R F^2 = F^2 \times (T - R) + R G^2$$

and

$$\begin{aligned}F^2 &= 3 T^2 \\ \therefore \frac{G^2}{F^2} &= \frac{T^3}{R F^2} - \frac{T-R}{R} \\ &= \frac{T^3}{3 R T^2} - \frac{T-R}{R} \\ &= \frac{T}{3 R} - \frac{T-R}{R} = \frac{3 R - 2 T}{3 R} \\ &= \frac{1}{3} \text{nearly}\end{aligned}$$

since R is $=$ T nearly.

Hence when $F = 180^\circ = \pi$

$$\gamma = G = \frac{\pi}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots \quad (1)$$

the angle between the apsides of the Locus in which the force is constant.

358. Ex. 2. Let $X' \propto \varrho^{n-3}$. Then as before

$$(T-x)^n = F^2 (T-x) + R G^2 - R F^2$$

and expanding and equating homologous terms

$$T^n = F^2 T + R G^2 - R F^2$$

and

$$F^2 = n T^{n-1}.$$

But since T nearly = R

$$T^{n-1} = G^2$$

$$\therefore \frac{G^2}{F^2} = \frac{1}{n}$$

and when F = π

$$\gamma = G = \frac{\pi}{\sqrt{n}}.$$

Thus when $n - 3 = 1$, we have

$$\gamma = \frac{\pi}{\sqrt{4}} = \frac{\pi}{2} = 90^\circ.$$

When $n - 3 = -1$, $n = 2$, and $\gamma = \frac{\pi}{\sqrt{2}} = 127^\circ.\ 16'.\ 45''.$

When $n - 3 = -\frac{11}{4}$, $n = \frac{1}{4}$, and $\gamma = 2\pi = 360^\circ$.

359. Let $X' \propto \frac{b\varrho^m \pm c\varrho^n}{\varrho^3}$. Then

$$b.(T-x)^m \pm c.(T-x)^n = F^2.(T-x) + R.(G^2 - F^2)$$

and expanding and equating homologous terms we get

$$b T^m \pm c T^n = F^2 (T - R) + R G^2$$

and

$$b m T^{m-1} \pm c n T^{n-1} = F^2.$$

But R being nearly = T, we have

$$b T^{m-1} \pm c T^{n-1} = G^2$$

$$\therefore \frac{G^2}{F^2} = \frac{b T^{m-1} \pm c T^{n-1}}{b m T^{m-1} \pm c n T^{n-1}} = \frac{b T^m \pm c T^n}{m b T^m \pm n c T^n}$$

which is more simply expressed by putting $T = 1$. Then we have

$$\frac{G^2}{F^2} = \frac{b + c}{m b \pm n c}$$

and when F = π

$$\gamma = G = \pi \sqrt{\frac{b \pm c}{m b \pm n c}}.$$

360. COR. 1. *Given the $\angle$ between the apsides to find the force.*

$$\text{Let } n : m :: 360^\circ : 2\gamma$$

$$:: 180^\circ = \pi : \gamma$$

$$\therefore \gamma = \frac{m}{n} \pi$$

But if $X' \propto \varrho^{p-3}$

$$\gamma = \frac{\pi}{\sqrt{p}}.$$

$$\therefore p = \frac{n^2}{m^2}$$

$$\therefore X' \propto \varrho^{\frac{n^2}{m^2} - 3}$$

Ex. 1. If n : m : : 1 : 1,

$$X' \propto \frac{1}{\varrho^2}$$

as in the ellipse about the focus.

2. If n : m : : 363 : 360

$$X' \propto \varrho^{\left(\frac{120}{121}\right)^2 - 3}$$

3. If n : m : : 1 : 2

$$X' \propto \frac{1}{\varrho^{\frac{11}{4}}}.$$

And so on.

$$\text{Again if } X' \propto \frac{1}{\varrho^3}$$

$$\gamma = \frac{\pi}{\sqrt{0}} = \infty$$

and the body having reached one apse can never reach another.

$$\text{If } X' \propto \frac{1}{\varrho^{3+q}}$$

$$\gamma = \frac{\pi}{\sqrt{-q}}$$

$\therefore$ the body never reaches another apse, and since the centrifugal force $\propto \frac{1}{\varrho^3}$, if the body depart from an apse and centrifugal force be $>$ centripetal force, then centrifugal is always $>$ centripetal force and the body will continue to ascend in infinitum.

Again if at an apse the centrifugal be $<$ the centripetal force, the centrifugal is always $<$ centripetal force and the body will descend to the center.

The same is true if $X' \propto \frac{1}{\varrho^3}$ and in all these cases, if

$$\text{centrifugal} = \text{centripetal}$$

the body describes a circle.

361. COR. 2. First let us compare the force $\frac{1}{A^2} - c\,A$, belonging to the moon's orbit, with

$$\frac{F^2}{A^2} + \frac{R\,G^2 - R\,F^2}{A^3}.$$

Since the moon's apse proceeds, (n m) is positive.

$\therefore -\mathrm{c\,A}$ does not correspond to n m and $\therefore \frac{1}{A^2}$ does not correspond to $\frac{F^2}{A^2}$.

Now

$$\frac{1}{A^2} - c\,A \propto \frac{A - c\,A^4}{A^3} \propto \frac{b\,A^m - c\,A^n}{A^3}$$

$$\therefore X' \propto A^{\frac{1-4c}{1-c}-3} \propto A^{\frac{F^2}{G^2}-3}$$

$$\therefore \frac{1-4c}{1-2} = \frac{F^2}{G^2}$$

$$\therefore \frac{F^2}{A^2} + \frac{R\,G^2 - R\,F^2}{A^3} = \frac{1-4c}{A^2} + \frac{3\,c\,R}{A^3}$$

$$\therefore \frac{F^2}{A^2} = \frac{1-4c}{A^2} \text{ and not } \frac{1}{A^2}$$

and

$$m\,n = \frac{3\,c\,R}{A^2}.$$

Hence also

$$\gamma = \pi \sqrt{\frac{1-c}{1-4c}}.\ \&c.\ \&c.\ \&c.$$

362. *To determine the angle between the apsides generally.*

Let

$$X \propto \frac{f(A)}{A^3}$$

f (A) meaning any function whatever of A. Then for Trajectories which are nearly circular, put

$$\frac{f(A)}{A^3} = \frac{F^2 A + R.(G^2 - F^2)}{A^3}$$

$$\therefore f.\,A = F^2 A + R\,(G^2 - F^2)$$

or

$$f.(T - x) = F^2 (T - x) + R\,(G^2 - F^2)$$

But expanding f (T — x) by Maclaurin's Theorem (32)

$$u = f(T - x) = U - U'x + U''\frac{x^2}{2} - \&c.$$

U, U′ &c. being the values of u, $\frac{d\,u}{d\,x}$, $\frac{d^2 u}{d\,x^2}$ &c.

when x = 0, and therefore independent of x. Hence comparing homologous terms (6) we have

$$U = F^2 T + R\,(G^2 - F^2)$$
$$U' = F^2$$

Also since R = T nearly

$$U = T G^2$$

$$\frac{G^2}{F^2} = \frac{U}{T . U'} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Hence when $F = \pi$, the angle between the apsides is

$$\left.\begin{aligned} \gamma = G &= \pi \sqrt{\frac{U}{T . U'}} \\ \text{or} \quad &= \pi \sqrt{\frac{U}{U'}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

making $T = 1$.

Ex. 1. Let $f(A) = b A^m + c A^n = u$

Then

$$\frac{d u}{d x} = m b A^{m-1} + n c A^{n-1}.$$

Hence since $A = T$ when $x = 0$

$$U = f T = b T^m + c T^n$$

$$U' = m b T^{m-1} + n c T^{n-1}$$

$$\therefore \frac{G^2}{F^2} = \frac{b T^m + c T^n}{m b T^m + n c T^n}$$

or

$$\frac{G^2}{F^2} = \frac{b + c}{m b + n c}$$

and

$$\gamma = \pi \sqrt{\frac{b + c}{m b + n c}}$$

as in 359.

Ex. 2. Let $f . (A) = b A^m + c A^n + e A^r + \&c.$

$$\therefore \frac{d u}{d x} = m b A^{m-1} + n c A^{n-1} + r e A^{r-1} + \&c.$$

$$\therefore U = b T^m + c T^n + e T^r + \&c.$$

and

$$T \times U' = m b T^m + n c T^n + r e T^r + \&c.$$

$$\therefore \frac{G^2}{F^2} = \frac{b T^m + c T^n + e T^r + \&c.}{m b T^m + n c T^n + r e T^r + \&c.}$$

or

$$= \frac{b + c + e + f + \&c.}{m b + n c + r e + s f + \&c.}$$

when $T = 1$.

Also

$$\gamma = \pi \sqrt{\frac{b + c + e \ldots\ldots}{m b + n c + r e + \ldots\ldots}}$$

Ex. 3. Let $\frac{f(A)}{A^3} = a^A = u$.

Here (17)

$$\frac{du}{dx} = A^2 a^A \times (3 + A\,l\,a)$$

Hence

$$U = T^2 a^T \times (3 + T\,l\,a)$$

$$T \times U' = T^3 a^T (3 + T\,l\,a)$$

$$\frac{G^2}{F^2} = \frac{1}{T \times (3 + T\,l\,a)}$$

and when $T = 1$

$$\frac{G^2}{F^2} = \frac{1}{3 + l\,a}$$

$$\therefore \gamma = \pi \sqrt{\frac{1}{3 + l\,a}}$$

Hence if $a = e$ the hyperbolic base, since $l\,e = 1$, we have

$$\gamma = \frac{\pi}{2}.$$

Ex. 4. Let $f(A) = e^A = u$.

Then

$$\frac{du}{dx} = e^A$$

$$\therefore U = e^T$$

and

$$T.U' = T\,e^T$$

$$\therefore \frac{G^2}{F^2} = \frac{1}{T}$$

$$\therefore \gamma = \pi.$$

Ex. 5. Let $\frac{f(A)}{A^3} = \sin. A$.

$$u = f(A) = A^3 \sin. A$$

$$\therefore U = T^3 \sin. T$$

and

$$\frac{du}{dx} = 3\,A^2 \sin. A + A^3 \cos. A$$

$$\therefore T\,U' = 3\,T^3 \sin. T + T^4 \cos. T$$

$$\therefore \frac{G^2}{F^2} = \frac{\sin. T}{3 \sin. T + T \cos. T}$$

$$\therefore \gamma = \pi \sqrt{\frac{\sin. T}{3 \sin. T + T \cos. T}}.$$

If $T = \frac{\pi}{4}$. Then

$$\gamma = \pi \sqrt{\frac{1}{3 + \frac{\pi}{4}}}.$$

363. *To prove that*

$$\frac{b A^m + c A^n}{A^3} = \frac{1}{b+c} . A^{\frac{mb+nc}{b+c} - 3}.$$

$$\begin{aligned} b A^m + c A^n &= b . (1-x)^m + c . (1-x)^n \\ &= b + c - (m b + n c) x + \&c. \\ &= \frac{1}{b+c}\left(1 - \frac{m b + n c}{b+c} x + \&c.\right) \\ &= \frac{1}{b+c} \times (1-x)^{\frac{mb+nc}{b+c}} \\ &= \frac{1}{b+c} . A^{\frac{mb+nc}{b+c}}. \end{aligned}$$

364. *To find the apsides when the excentricity is infinitely great.*

Make

$2q : \sqrt{(n+1)} : :$ velocity in the curve : velocity in the circle of the same distance a.

Then (306) it easily appears that when $F \propto \varrho^n$

$$d\theta = \frac{q a^{\frac{n+3}{2}} d\varrho}{\varrho \sqrt{(a^{n+1} - \varrho^{n+1}) \varrho^2 - q^2 a^{n+1} (a^2 - \varrho^2)}}$$

and

$$\frac{d\varrho}{d\theta} = 0$$

gives the equation to the apsides, viz.

$$(a^{n+1} - \varrho^{n+1}) \varrho^2 - q^2 a^{n+1} (a^2 - \varrho^2) = 0$$

whose roots are

a (and — a when n is odd) and a positive and negative quantity (and when n is odd another negative quantity).

Now when $q = 0$

$$(a^{n+1} - \varrho^{n+1}) \varrho^2 = 0$$

two of whose roots are 0, 0, and the roots above-mentioned consequently arise from q, which will be very small when q is.

Again since

$$1 - \frac{\varrho^{n+1}}{a^{n+1}} - \frac{a^2 q^2}{\varrho^2} + q^2 = 0$$

when q and ϱ are both very small

$$1 - \frac{a^2 q^2}{\varrho^2} = 0$$

and

$$\varrho = \pm a\,q.$$

∴ the lower apsidal distance is a q.

A nearer approximation is

$$\varrho = \pm \frac{a\,q}{\sqrt{(1+q^2)}}.$$

Hence

$$d\,\theta = \frac{q\,a^{\frac{n+3}{2}}\,d\,\varrho}{\varrho\sqrt{(\varrho^2 - a^2 q^2 + \beta)} \times Q}$$

where β contains q^4 &c. &c., and this must be integrated from $\varrho = b$ to $\varrho = a$ ($b = a\,q$).

But since in the variation of ϱ from b to c, Q may be considered constant, we get

$$\theta = \sec.^{-1}.\frac{\varrho}{a\,q} + C = \sec.^{-1}.\frac{c}{a\,q}.$$

and

$$\gamma = \frac{\pi}{2}, \frac{3\,\pi}{2}, \frac{5\,\pi}{2}, \text{ \&c. ultimately}$$

the apsidal distances required.

Next let

$$F \propto \frac{1}{\varrho^n} \text{ and } = \frac{f\,a^n}{\varrho^n}.$$

Then again, make

v : v in a circle of the same distance : : $q\sqrt{2} : \sqrt{(n-1)}$

and we get (306)

$$d\,\theta = \frac{q\,a\,d\,\varrho}{\varrho\sqrt{a^{n-1}\varrho^{3-n} - (1-q^2)\varrho^2 - a^2 q^2}}$$

and for the apsidal distances

$$1 - \frac{\varrho^{n-1}}{a^{n-1}} + \frac{q^2 \varrho^{n-1}}{a^{n-1}} - \frac{q^2 a^{3-n}}{\varrho^{3-n}} = 0$$

which gives ($n > 1$ and < 3)

$$\varrho = a\,q^{\frac{2}{3-n}}$$

Hence

$$\theta = \int \frac{a\,q\,d\,\varrho}{\varrho\sqrt{(\varrho^{3-n} - q^2 a^{3-n} + \beta)} \times Q}$$

$$= \frac{1}{\sqrt{Q}} \cdot \int \frac{a\,q\,d\,\varrho}{\varrho\sqrt{(\varrho^{3-n} - q^2 a^{3-n}}}$$

and

$$\gamma = \frac{2}{3-n}\,\text{sec.}^{-1}\frac{c^{\frac{3-n}{2}}}{q\,a^{\frac{3-n}{2}}} = \frac{\pi}{3-n} = \frac{3\pi}{3-n}, \text{ \&c.}$$

Hence, the orbit being indefinitely excentric, when

$F \propto \varrho$ . . . . . we have . . . . . $\gamma = \frac{\pi}{2}$

for

$F \propto \frac{1}{\text{any number} < 1}$ . . . . $\gamma = \frac{\pi}{2}$

$F \propto \frac{1}{\varrho}$ . . . . $\gamma = \frac{\pi}{2}$

$F \propto \frac{1}{\varrho^{\text{number between 1 and 2}}}$ . . . . . $\gamma > \frac{\pi}{2} < \pi$

$F \propto \frac{1}{\varrho^{>2<3}}$ . . . . $\gamma > \pi$.

But by the principles of this 9th Section when the excentricity is indefinitely small, and $F \propto \varrho^{n}$

$$\gamma = \frac{\pi}{\sqrt{(n+3)}}$$

(see 358), and when

$$F \propto \frac{1}{\varrho^{n}}$$

$$\gamma = \frac{\pi}{\sqrt{(3-n)}}$$

Wherefore when n is > 1

γ increases as the excentricity from

$$\frac{\pi}{\sqrt{(3+n)}} \text{ to } \frac{\pi}{2}.$$

When $F \propto \varrho$

$\gamma = \frac{\pi}{2}$ is the same for all excentricities.

When $F \propto \varrho^{\pm\frac{p}{q}}$

γ decreases as the excentricity increases from

$$\frac{\pi}{\sqrt{(3-n)}} \text{ to } \frac{\pi}{2}$$

which is also true for $F \propto \frac{1}{\varrho}$.

When $F \propto \frac{1}{\varrho^{>1<2}}$

γ decreases as the excentricity increases from

$$\frac{\pi}{\sqrt{(3-n)}} \text{ to } \frac{\pi}{3-n}.$$

When $F \propto \frac{1}{\varrho^2}$

$$\gamma = \pi.$$

When $F \propto \frac{1}{\varrho^{>2<3}}$

γ increases with the excentricity from

$$\frac{\pi}{\sqrt{(3-n)}} \text{ to } \frac{\pi}{3-n}.$$

If the above concise view of the method of finding the apsides in this particular case, the opposite of the one in the text, should prove obscure; the student is referred to the original paper from which it is drawn, viz. a very able one in the Cambridge Philosophical Transactions, Vol. I, Part I, p. 179, by Mr. Whewell.

365. We shall terminate our remarks upon this Section by a brief discussion of the general apsidal equations, or rather a recapitulation of results—the details being developed in Leybourne's Mathematical Repository,—by Mr. Dawson of Sedburgh.

It will have been seen that the equation of the apsides is of the form

$$x^n - A x^m - B = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the equation of Limits to which is (see Wood's Algeb.)

$$n x^{n-1} - m A x^{m-1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and gives

$$x = \left(\frac{m}{n} A\right)^{\frac{1}{n-m}}.$$

If n and m are even and A positive, x has two values, and the number of real roots cannot exceed 4 in that case.

Multiply (1) by n and (2) by x and then we have

$$(m-n) A x^m - n B = 0$$

which gives

$$x = \left(\frac{n}{m-n}\right)^{\frac{1}{m}} \left(\frac{B}{A}\right)^{\frac{1}{m}}$$

and this will give two other limits if A, B be positive and m even; and if (1) have two real roots they must each $= x$.

If m, n be even and B, A positive, there will be two pairs of equal roots. Make them so and we get

$$\frac{(m-n)^{n-m}}{n^{n-m}} A^{n} - \left(\frac{n}{m}\right)^{m} B^{n-m} = 0$$

which will give the number of real roots.

(1). If n be even and B positive there are two real roots.

(2). If n be even, m odd, and B negative and (M), the coefficient to A^{n}, negative, there are two; otherwise none.

(3). If n, m, be even, A, B, negative, there are no real roots.

(4). If m, n be even, B negative, and A positive, and (M) positive there are four real roots; otherwise none.

(5). If m, n be odd, and (M) positive there will be three or one real.

(6). If m be even, n odd, and A, B have the same sign, there will be but one.

(7). If m be even, n odd, and A, B have different signs, and M's sign differs from B's, there will be three or only one.

(8). If

$$x^{n} + A n^{m} - B = 0$$

then

$$\left(\frac{m-n}{n}\right)^{n-m} A^{n}.$$

is positive, and it must be > B, and the whole must be positive.

If

$$x^{n} - A x^{m} + B = 0$$

the result is negative.

SECTION X.

366. PROP. XLVI. The shortest line that can be drawn to a plane from a given point is the perpendicular let fall upon it. For since Q C S = right ∠, any line Q S which subtends it must be > than either of the others in the same triangle, or S C is < than any other S C.

A familiar application of this proposition is this:

367. *Let* S Q *be a sling with a body* Q *at the end of it, and by the hand* S *let it be whirled so as to describe a right cone whose altitude is* S C, *and base the circle whose radius is* Q C; *required the time of a revolution.*

Let S C = h, S Q = l, Q C = r = $\sqrt{l^2 - h^2}$.

Then if F denote the resolved part of the tension S Q in the direction Q C, or that part which would cause the body to describe the circle P Q, and gravity be denoted by 1, we have

$$F : 1 :: r : h$$

$$\therefore F = \frac{r}{h}.$$

But by 134, or Prop. IV,

$$F \times P^2 = \frac{4\pi^2 \times r}{g} = \frac{r}{h} \times P^2$$

$$\therefore P = 2\pi\sqrt{\frac{h}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the time required.

If the time of revolution (P) be *observed*, then h may be hence obtained.

If a body were to revolve round a circle in a paraboloidal surface, whose axis is vertical, then the reaction of the surface in the direction of the normal will correspond to the tension of the string, and the subnormal, which is constant, will represent h. Consequently the times of all such revolutions is constant for every such circle.

368. Prop. XLVII. When the excentricity of the ellipse is indefinitely diminished it becomes a straight line in the limit, &c. &c. &c.

369. Scholium. In these cases it is sufficient to consider the motion in the generating curves.]

Since the surface is supposed perfectly smooth, whilst the body moves through the generating curve, the surface, always in contact with the body, may revolve about the axis of the curve with any velocity whatever, without deranging in the least the motion of the body; and thus by adjusting the angular velocity of the surface, the body may be made to trace any proposed path on the surface.

If the surface were not perfectly smooth the friction would give the body a tangential velocity, and thence a centrifugal force, which would cause a departure from both the curve and surface, unless opposed by their material; and even then in consequence of the resolved pressure a rise or fall in the surface.

Hence it is clear that the time of describing any portion of a path in a surface of revolution, is equal to the time of describing the corresponding portion of the generating curve.

Thus when the force is in the center of a sphere, and whilst this force causes the body to describe a fixed great-circle, the sphere itself revolves with a uniform angular velocity, the path described by the body on the surface of the sphere will be the Spiral of Pappus.

370. PROP. XLVIII and XLIX. *In the Epicycloid and Hypocycloid,*

$$s : 2\,\text{vers.}\,\frac{s'}{2} :: 2\,(R \pm r) : R$$

where s *is any arc of the curve,* s′ *the corresponding one of the wheel, and* R *the radius of the globe and* r *that of the wheel, the* + *sign being used for the former and* — *in the Hypocycloid.* (See Jesuits' notes.)

OTHERWISE.

If p be the perpendicular let fall from C upon the tangent V P, we have from similar triangles in the Epicycloid and Hypocycloid

$$PY : CB :: VY : VC$$

or

$$\varrho^2 - p^2 : R^2 :: (R \pm 2\,r)^2 - p^2 : (R \pm 2\,r)^2$$

which gives

$$p^2 = (R \pm 2\,r)^2 \times \frac{\varrho^2 - R^2}{(R \pm 2\,r)^2 - R^2} \quad . \quad . \quad . \quad . \quad (1)$$

Now from the incremental figure of a curve we have generally

$$\frac{d\,s}{d\,\varrho} = \frac{\varrho}{\sqrt{(\varrho^2 - p^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But

$$\varrho^2 - p^2 = \frac{R^2}{(R \pm 2\,r)^2 - R^2} \times \{(R \pm 2\,r)^2 - \varrho^2\}$$

$$\therefore d\,s = \frac{2\,\sqrt{r^2 \pm R\,r}}{R} \times \frac{\varrho\,d\,\varrho}{\sqrt{(R \pm 2\,r)^2 - \varrho^2}}$$

and integrating from

$$s = 0, \text{ when } \varrho = R$$

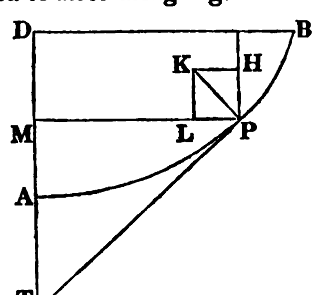
we get

$$s = \frac{2\,\sqrt{r^2 \pm R\,r}}{R} \times \{\sqrt{(R \pm 2\,r)^2 - R^2} - \sqrt{(R \pm 2\,r)^2 - \varrho^2}\}$$

which is easily transformed to the proportion enunciated.

The subsequent propositions of this section shall now be headed by a succinct view of the analytical method of treating the same subject.

371. Generally, *A body being constrained to move along a given curve by known forces, required its velocity.*



Let the body P move along the curve P A, referred to the coordinates x, y originating in A; and let the forces be resolved into others which shall act parallel to x, y and call the respective aggregates X, Y. Besides these we have to consider the reaction (R) of the

curve along the normal P K, which being resolved into the same directions gives (d s, being the element of the curve)

$$R\frac{dx}{ds}, \text{ and } R\frac{dy}{ds}.$$

Hence the whole forces along x and y are (see 46)

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X + R\frac{dy}{ds}\\ \frac{d^2y}{dt^2} &= Y - R\frac{dx}{ds}\end{aligned}\right\}$$

Again, eliminating R, we get

$$\frac{2dxd^2x + 2dyd^2y}{dt^2} = 2Xdx + 2Ydy.$$

and

$$\frac{dx^2 + dy^2}{dt^2} = 2\int(Xdx + Ydy)$$

But

$$v^2 = \frac{ds^2}{dt^2} = \frac{dx^2 + dy^2}{dt^2} \;(46)$$

$$\therefore v^2 = 2\int(Xdx + Ydy) \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

Hence it appears that *The velocity is independent of the reaction of the curve.*

372. If the force be constant and in parallel lines, such as gravity, and x be vertical; then

$$X = -g$$

and

$$Y = 0$$

and we have

$$\begin{aligned}v^2 &= 2\int - g\,dx\\ &= 2g(c - x)\\ &= 2g(h - x)\end{aligned}$$

h being the value of x, when $v = 0$; and the height from which it begins to fall.

373. *To determine the motion in a common cycloid, when the force is gravity.*

The equation to the curve A P is

$$dy = dx\sqrt{\frac{2r - x}{x}}$$

r being the radius of the generating circle.

$$\therefore ds = dx\sqrt{\frac{2r}{x}}$$

and

$$dt = -\frac{ds}{\sqrt{2g}.\sqrt{(h-x)}} = \sqrt{\frac{r}{g}} \times \frac{dx}{\sqrt{(hx-x^2)}}$$

$$\therefore t = C - \sqrt{\frac{r}{g}}\,\text{vers.}^{-1}\frac{2x}{h}\ (86)$$

$$= \sqrt{\frac{r}{g}}\left\{\pi - \text{vers.}^{-1}\frac{2x}{h}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

t being $= 0$, when $x = h$.

Hence the whole time of descent to the lowest point is

$$\frac{T}{2} = \pi\sqrt{\frac{r}{g}}$$

which also gives the time of an oscillation.

374. *Required the time of an oscillation in a small circular arc.*

Here

$$y = \sqrt{(2rx - x^2)}$$

r being the radius of the circle, and

$$ds = \frac{r\,dx}{\sqrt{(2rx-x^2)}}$$

$$\therefore dt = \frac{ds}{\sqrt{2g}\sqrt{(h-x)}}$$

$$= -\frac{r}{\sqrt{2g}} \times \frac{dx}{\sqrt{\{h-x)(2rx-x^2)\}}}$$

$$= -\frac{r}{\sqrt{2g}} \times \frac{dx}{\sqrt{\{(hx-x^2)(2r-x)\}}}$$

to integrate which, put

$$\theta = \sin.^{-1}\sqrt{\frac{x}{h}},$$

$$\therefore d\theta = \frac{dx}{2\sqrt{(hx-x^2)}}$$

and since

$$\sqrt{\frac{x}{h}} = \sin.\theta$$

$$x = h\sin.^2\theta,\ 2r - x = 2r - h\sin.^2\theta$$

$$= 2r(1-\delta^2\sin.^2\theta),\ \delta^2 \text{ being put} = \frac{h}{2r}$$

$$\therefore dt = -\sqrt{\frac{r}{g}} \times \frac{d\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}}.$$

Now since the circular arc is small, h is small; and therefore δ is so. And by expanding the denominator we get

$$\int\frac{d\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}} = \int d\theta\left\{1 + \frac{\delta^2}{2}\sin.^2\theta + \frac{1.3}{2.4}\delta^4\sin.^4\theta + \&c.\right\}$$

and integrating by parts or by the formula

$$\int d\,\theta\,.\sin.^{m}\theta = -\frac{1}{m}\cos.\theta\sin.^{m-1}\theta + \frac{m-1}{m}\int d\,\theta\sin.^{m-2}\theta$$

and taking it from

$$\theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

we get

$$\int_{\prime} d\,\theta\sin.^{m}\theta = \frac{m-1}{m}\int_{\prime} d\,\theta\sin.^{m-2}\theta$$

the accented $\int_{\prime}$ denoting the Definite Integration from $\theta = 0$, to $\theta = \frac{\pi}{2}$.

In like manner

$$\int_{\prime} d\,\theta\sin.^{m-2}\theta = \frac{m-3}{m-2}\int_{\prime} d\,\theta\sin.^{m-4}\theta$$

and so on to

$$\int_{\prime} d\,\theta\sin.^{2}\theta = \tfrac{1}{2}\int_{\prime} d\,\theta = \tfrac{1}{2}\,.\,\frac{\pi}{2}\,.$$

Hence

$$\int_{\prime} d\,\theta\sin.^{m}\theta = \frac{(m-1)\,(m-3)\,\ldots\ldots 1}{m\,(m-2)\,\ldots\ldots\ldots 2} \times \frac{\pi}{2}$$

and

$$\int_{\prime}\frac{d\,\theta}{\sqrt{(1-\delta^2\sin.^2\theta}} \text{ from } \left.\begin{matrix}\theta = 0\\ \theta = \frac{\pi}{2}\end{matrix}\right\}$$

is the same as

$$\int_{\prime} - \frac{d\,\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}} \text{ from } \left.\begin{matrix}\theta = \frac{\pi}{2}\\ \theta = 0\end{matrix}\right\}$$

whence then

$$t = \frac{\pi}{2}\sqrt{\frac{r}{g}}\left\{1 + \left(\frac{\delta}{2}\right)^2 + \left(\frac{1.\,3\,\delta^2}{2.\,4}\right)^2 + \left(\frac{1.\,3.\,5.\,\delta^2}{2.\,4.\,6}\right)^3 + \&c.\right\} \quad . . (2)$$

and taking the first term only as an approximate value

$$t = \frac{\pi}{2}\sqrt{\frac{r}{g}} \quad . . . . (3)$$

which equals the time down a cycloidal arc whose radius is $\frac{r}{4}$.

If we take two terms we have

$$t = \frac{\pi}{2}\sqrt{\frac{r}{g}}\left(1 + \frac{\delta^2}{4}\right)$$

$$= \frac{\pi}{2}\sqrt{\frac{r}{g}}\left(1 + \frac{h}{8\,r}\right) \quad . . . . (4)$$

375. To determine the velocity and time in a Hypocycloid, the force tending to the center of the globe and $\propto \varrho$.

By (370)

the equation to the Hypocycloid is

$$p^2 = \frac{R^2 - \varrho^2}{R^2 - (2r - R)^2} \times (2r - R)^2$$
$$= \frac{R^2 - \varrho^2}{R^2 - D^2} \times D^2$$

by hypothesis.

Now calling the force tending to the center F, we have

$$X = -F \times \frac{x}{\varrho}, \; Y = -F \times \frac{y}{\varrho}$$

$$\therefore \int (X dx + Y dy) = -\int F \frac{x dx + y dy}{r}$$
$$= -\int F d\varrho$$
$$\therefore v^2 = C - 2\int F d\varrho \quad \ldots \ldots \ldots \quad (1)$$

But by the supposition

$$F = \mu \varrho$$
$$\therefore v^2 = \mu (h^2 - \varrho^2) \quad \ldots \ldots \ldots \quad (2)$$

Hence

$$dt = \frac{ds}{v} = -\frac{\sqrt{R^2 - D^2}}{R\sqrt{\mu}} \times \frac{\varrho d\varrho}{\sqrt{\{(\varrho^2 - D^2)(h^2 - \varrho^2)\}}}.$$

To integrate it, put

$$\varrho^2 - D^2 = u^2$$
$$\therefore \frac{\varrho d\varrho}{\sqrt{(\varrho^2 - D^2)}} = du$$
$$h^2 - \varrho^2 = h^2 - D^2 - u^2$$

and

$$dt = -\frac{\sqrt{R^2 - D^2}}{R\sqrt{\mu}} \; \frac{du}{\sqrt{(h^2 - D^2 - u^2)}}$$
$$\therefore t = \frac{\sqrt{(R^2 - D^2)}}{R\sqrt{\mu}} \cos.^{-1} \sqrt{\frac{\varrho^2 - D^2}{h^2 - D^2}}.$$

Hence making $\varrho = D$, we have

$$\frac{\text{Oscill.}}{2} = \frac{\pi}{2}\sqrt{\frac{R^2 - D^2}{R^2 \mu}} \quad \ldots \ldots \ldots \quad (3)$$

376. Since h does not enter the above expression the descents are Isochronous.

We also have it in another form, viz.

$$\frac{T}{2} = \pi \sqrt{\left(\frac{r}{R\mu} - \frac{r^2}{R^2 \mu}\right)}.$$

If R μ = g or force of gravity and R be large compared with b,

$$\frac{T}{2} = \pi \sqrt{\frac{r}{g}}$$

the same as in the common cycloid.

377. *Required to find the value of the reaction* R, *when a body is constrained to move along a given curve.*

As before (46)

$$\frac{d^2 x}{d t^2} = X + R \frac{d y}{d x}$$

$$\frac{d^2 y}{d t^2} = Y - R \frac{d x}{d s}.$$

Hence

$$\frac{d y d^2 x - d x d^2 y}{d t^2} = X d y - Y d x + R d s$$

$$\therefore R = -\frac{X d y - Y d x}{d s} + \frac{d y d^2 x - d x d^2 y}{d t^2 d s}.$$

But if r be the radius of curvature, we have (74)

$$r = \frac{d s^3}{d y d^2 x - d x d^2 y}.$$

Hence

$$R = \frac{Y d x - X d y}{d s} + \frac{d s^2}{r d t^2} \quad \ldots \quad (1)$$

Another expression is

$$\left.\begin{aligned} R &= \frac{Y d x - X d y}{d s} + \frac{v^2}{r} \\ \text{or} \quad &= \frac{Y d x - X d y}{d s} + \varphi \end{aligned}\right\} \quad \ldots \quad (2)$$

φ being the centrifugal force.

If the body be acted on by gravity only

$$\left.\begin{aligned} R &= \frac{g d y}{d s} + \frac{d s^2}{r d t^2} \\ \text{or} \quad &= \frac{g d y}{d s} + \frac{v^2}{r} \\ \text{or} \quad &= \frac{g d y}{d s} + \varphi \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

If the body be moved by a constant force in the origin of x, y, we have

$$Y d x - X d y = F \frac{x d y - y d x}{\varrho}$$

$$= F \varrho d \theta.$$

for

$$x\,dy - y\,dx = \varrho^2\,d\theta$$

$$\left.\begin{aligned}\therefore R &= \frac{F\,\varrho\,d\theta}{ds} + \frac{ds^2}{r\,dt^2}\\ \text{or}\quad &= \frac{F\,\varrho\,d\theta}{ds} + \frac{v^2}{r}\\ \text{or}\quad &= \frac{F\,\varrho\,d\theta}{ds} + \varphi\end{aligned}\right\}\quad\ldots\ldots\ (4)$$

378. *To find the tension of the string in the oscillation of a common cycloid.*

Here

$$R = g\frac{dy}{ds} + \frac{ds^2}{r\,dt^2}$$

but

$$dy = dx\sqrt{\frac{2a - x}{x}}$$

$$ds = dx\sqrt{\frac{2a}{x}}.$$

$$\frac{dy}{ds} = \sqrt{\frac{2a - x}{2a}}$$

and

$$r = 2\sqrt{2a}\sqrt{(2a - x)}$$

$$\frac{ds^2}{dt^2} = 2g(h - x)$$

$$\therefore R = g\sqrt{\frac{2a - x}{2a}} + \frac{g(h - x)}{\sqrt{2a}\sqrt{(2a - x)}}$$

$$= g \cdot \frac{2a + h - 2x}{\sqrt{(4a^2 - 2ax)}}.$$

When $x = h$

$$R = g \cdot \frac{2a - h}{\sqrt{(4a^2 - 2ah)}} = g\frac{\sqrt{(2a - h)}}{\sqrt{(2a)}}.$$

When $x = 0$

$$R = g \cdot \frac{2a + h}{2a} = g\left(1 + \frac{h}{2a}\right).$$

When moreover $h = 2a$, the pressure at A the lowest point is $= 2g$.

379. *To find the tension when the body oscillates in a circular arc by gravity.*

Here

$$d\,y = \frac{(c - x)\,d\,x}{\sqrt{(2\,c\,x - x^2)}}$$

$$d\,s = \frac{c\,d\,x}{\sqrt{(2\,c\,x - x^2)}}$$

$$\frac{d\,y}{d\,x} = \frac{c - x}{c}$$

$$r = c^{\cdot}$$

$$\frac{d\,s^2}{d\,t^2} = 2\,g\,(h - x)$$

$$R = g \cdot \frac{c - x}{c} + \frac{2\,g\,(h - x)}{c}$$

$$= g \cdot \frac{c + 2\,h - 3\,x}{c}.$$

When $x = 0$

$$R = g \cdot \frac{c + 2\,h}{c}$$

$$= 3\,g \text{ or } h = c.$$

If it fall through the whole semicircle from the highest point

$$h = 2\,c,$$

and

$$R = 5\,g$$

or the tension at the lowest point is five times the weight.

When this tension $= 0$,

$$c + 2\,h - 3\,x = 0, \text{ or } x = \frac{c + 2\,h}{3}.$$

A body moving along a curve whose plane is vertical will quit it when

$$R = 0$$

that is when

$$x = \frac{c + 2\,h}{3}$$

and then proceed to describe a parabola.

380. *To find the motion of a body upon a surface of revolution, when acted on by forces in a plane passing through the axis.*

Referring the surface to three rectangular axes x, y, z, one of which (z) is the axis of revolution, another is also situated in the plane of forces, and the third perpendicular to the other two.

Let the forces which act in the plane be resolved into two, one parallel to the axis of revolution Z, and the other F, into the direction of the radius-vector, projected upon the plane perpendicular to this axis. Then,

calling this projected radius ϱ, and resolving the reaction R (which also takes place in the same plane as the forces) into the same directions, these components are

$$R\frac{dz}{ds}$$

$$R\frac{d\varrho}{ds}$$

supposing $ds = \surd(dz^2 + d\varrho^2)$ and the whole force in the direction of ϱ' is

$$F + R\frac{dz}{ds}$$

and resolving this again parallel to x and y, we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\left(F + R\frac{dz}{ds}\right)\frac{x}{\varrho} \\ \frac{d^2y}{dt^2} &= -\left(F + R\frac{dz}{ds}\right)\frac{y}{\varrho} \\ \text{and} \quad \frac{d^2z}{dt^2} &= -Z + R\frac{d\varrho}{ds} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

Hence we get

$$\frac{xd^2y - yd^2x}{dt^2} = 0 = d.\frac{xdy - ydx}{dt} \quad \ldots\ldots \quad (2)$$

and

$$\frac{dxd^2x + dyd^2y + dzd^2z}{dt^2} = -F.\frac{xdx + ydy}{\varrho}$$

$$-Zdz - R.\frac{dz}{ds}\left\{\frac{xdx + ydy}{\varrho} - \frac{d\varrho dz}{ds}\right\} \quad \ldots \quad (3)$$

Which, since

$$\frac{xdx + ydy}{\varrho} = d\varrho$$

$$d\left(\frac{dx^2 + dy^2 + dx^2}{dt^2}\right) = -2Fd\varrho - 2Zdz.$$

Again

$$\frac{dz^2}{dt^2} = \frac{dz^2}{d\varrho^2}.\frac{d\varrho^2}{dt^2},$$

and from the nature of the section of the surface made by a plane passing through the axis and body, $\frac{dz}{d\varrho}$ is known in terms of ϱ. Let therefore

$$\frac{dz}{d\varrho} = p$$

and we have

$$\frac{d z^2}{d t^2} = p^2 \frac{d \varrho^2}{d t^2}.$$

Also let the angle corresponding to ϱ be θ, then

$$x d y - y d x = \varrho^2 d \theta$$

and

$$d x^2 + d y^2 = d \varrho^2 + \varrho^2 d \theta^2,$$

and substituting the equations (2) and (3) become

$$d . \frac{\varrho^2 d \theta}{d t} = 0$$

$$d \left\{ \frac{d \varrho^2}{d t^2} + \frac{\varrho^2 d \theta^2}{d t^2} + p^2 \frac{d \varrho^2}{d t^2} \right\} = -2 F d \varrho - 2 Z d z.$$

Integrating the first we have

$$\varrho^2 d \theta = h d t$$

h being the arbitrary constant.

or

$$d t = \frac{\varrho^2 d \theta^2}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The second can be integrated when

$$-2 F d \varrho - 2 Z d z$$

is integrable. Now if for F, Z, z we substitute their values in terms of ϱ, the expression will become a function of ϱ and its integral will be also a function of ϱ. Let therefore

$$\int (F d \varrho + Z d z) = Q$$

and we get

$$\frac{d \varrho^2}{d t^2} + \frac{\varrho^2 d \theta^2}{d t^2} + p^2 . \frac{d \varrho^2}{d t^2} = C - 2 Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which gives, putting for d t its value

$$(1 + p^2) \frac{h^2 d \varrho^2}{\varrho^4 d \theta^2} + \frac{h^2}{\varrho^2} = C - 2 Q$$

$$\therefore d \theta = \frac{\sqrt{(1 + p^2)} h d \varrho}{\sqrt{\{(c - 2 Q) \varrho^2 - h^2\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence also

$$d t = \frac{\sqrt{(1 + p^2)} . \varrho d \varrho}{\sqrt{\{(c - 2 Q) \varrho^2 - h^2\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the force be always parallel to the axis, we have

$$F = 0$$

and if also Z be a constant force, or if

$$Z = g$$

we then have

$$Q = \int Z d z = g z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Z being to be expressed in terms of ϱ.

381. *To find under what circumstances a body will describe a circle on a surface of revolution.*

For this purpose it must always move in a plane perpendicular to the axis of revolution; ϱ, z will be constant; also (Prop. IV)

$$\varrho \cos.\theta = x$$

$$\therefore \frac{d^2 x}{d t^2} = -\frac{\varrho \cos.\theta\, d\theta^2}{d t^2}$$

Also

$$v = \frac{\varrho\, d\theta}{d t}$$

$$\therefore \frac{d^2 x}{d t^2} = -\frac{v^2 \cos.\theta}{\varrho}.$$

Hence as in the last art.

$$\left.\begin{aligned}\frac{v^2}{\varrho} &= F + R\frac{d z}{d s}\\ 0 &= -Z + R\frac{d\varrho}{d s}\end{aligned}\right\}$$

$$\therefore \frac{v^2}{\varrho} = F + Z\frac{d z}{d\varrho} \quad \ldots\ldots\ldots\ldots \quad (1)$$

If the force be gravity acting vertically along z, we have

$$Z = g$$

$$\frac{v^2}{\varrho} = g\frac{d z}{d\varrho}.$$

Hence may be found the time of revolution of a Conical Pendulum. (See also 367.)

382. To determine the motion of a body moving so as not to describe a circle, when acted on by gravity.

Here

$$Q = g z$$

and

$$C - 2Q = 2g.(k - z)$$

k being an arbitrary quantity.

Also

$$\varrho^2 = 2 r z - z^2$$

z being measured from the surface.

$$\therefore \varrho\, d\varrho = (r - z)\, d z$$

and

$$1 + p^2 = 1 + \frac{\varrho^2}{(r-z)^2} = \frac{r^2}{(r-z)^2}.$$

Hence (380)

$$d\,t = \frac{\sqrt{(1+p^2)}\,\varrho\,d\,\varrho}{\sqrt{\{2\,g\,(k-z)\,.\,(2\,r\,z-z^2)-h^2\}}}.$$

In order that

$$\frac{d\,z}{d\,t} = 0$$

the denominator of the above must be put $= 0$; i. e.

$$2\,g\,(k-z)\,(2\,r\,z-z^2)-h^2 = 0$$

or

$$z^3-(k+2\,r)\,z^2+2\,k\,r\,z-\frac{h^2}{2\,g} = 0$$

which has two possible roots; because as the body moves, it will reach one highest and one lowest point, and therefore two places when

$$\frac{d\,z}{d\,t} = 0.$$

Hence the equation has also a third root. Suppose these roots to be

$$\alpha,\ \beta,\ \gamma$$

where α is the greatest value of z, and β the least, which occur during the body's motion.

Hence

$$d\,t = \frac{r\,d\,z}{\sqrt{(2\,g)}\,\sqrt{\{(\alpha-z)\,.\,(z-\beta)\,(\gamma-z)}}.$$

To integrate which let

$$\theta = \sin.^{-1}\sqrt{\frac{z-\beta}{\alpha-\beta}}.$$

Then

$$d\,\theta = \frac{d\,z}{2\sqrt{\{(z-\beta).(\alpha-\beta)\}}\sqrt{\left\{1-\frac{z-\beta}{\alpha-\beta}\right\}}}$$

$$= \frac{d\,z}{2\sqrt{\{(\alpha-z)\,(z-\beta)\}}}$$

Also

$$\sin.^2\theta = \frac{z-\beta}{\alpha-\beta}$$

$$\therefore z = \beta+(\alpha-\beta)\sin.^2\theta$$

and

$$\gamma-z = \gamma-\{\beta+(\alpha-\beta)\sin.^2\theta\}$$
$$= (\gamma-\beta)\,\{1-\delta^2\sin.^2\theta\},$$

if

$$\delta = \sqrt{\frac{\alpha-\beta}{\gamma-\beta}}$$

Z being to be expressed

381. *To find under ... surface of revolution.*

For this purpose ... axis of revolution ...

$$\frac{g r d\theta}{\sqrt{2g.(\gamma-\beta)}.\sqrt{\{1-\delta^2 \sin.^2\theta\}}}$$

which is to be integrated from z = β to z = α; that is from θ = 0 to θ = $\frac{\pi}{2}$

this expanded in the same way as in 374 gives

$$t = \frac{2r}{\sqrt{2g(\gamma-\beta)}} \times \left\{1+\left(\frac{\delta}{2}\right)^2+\left(\frac{1.3\,\delta^2}{2.4}\right)^2+\&c.\right\}\frac{\pi}{2}$$

which is the time of a whole oscillation from the least to the greatest distance.

Also

$$d\theta = \frac{h\,d\,t}{\rho^2} = \frac{h\,d\,t}{2\,r\,z - z^2}$$

and θ is hence known in terms of z.

383. *A body acted on by gravity moves on a surface of revolution whose axis is vertical: when its path is nearly circular, it is required to find the angle between the apsides of the path projected in the plane of* x, y.

In this case

$$\int Z\,d\,z = g\,z = Q$$

and if at an apse

$$\rho = a,\ z = k$$

we have

$$(C - 2gk)\,a^2 - h^2 = 0$$

$$\therefore C = \frac{h^2}{a^2} + 2gk.$$

Hence (380)

$$d\theta = \frac{\sqrt{(1+p^2)}\,h\,d\rho}{\rho\sqrt{\left\{2g(k-z)\rho^2 - \frac{h^2(a^2-\rho^2)}{a^2}\right\}}}$$

$$= \frac{\sqrt{(1+p^2)}\,\frac{h\,d\rho}{\rho^2}}{\sqrt{\left\{2g(k-z) - h^2\left(\frac{1}{\rho^2}-\frac{1}{a^2}\right)\right\}}}$$

Let $\frac{1}{\rho} = \frac{1}{a} + \omega$

$$\therefore -\frac{d\rho}{\rho^2} = d\omega$$

$$\therefore d\theta = -\frac{\sqrt{(1+p^2)}\,h\,d\omega}{\sqrt{\left\{2g(k-z) - h^2\left(\frac{1}{\rho^2}-\frac{1}{a^2}\right)\right\}}}$$

$$\therefore \frac{d\,\omega^2}{d\,\theta^2} = \frac{2\,g\,(k - z) - h^2\left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)}{h^2\,(1 + p^2)}.$$

It is requisite to express the right-hand side of this equation in terms of ω

Now since at an apse we have

$$\omega = 0,\ z = k, \text{ and } \varrho = a$$

we have generally

$$z = k + \frac{d\,z}{d\,\omega}\,\omega + \frac{d^2\,z}{d\,\omega^2}\cdot\frac{\omega^2}{1.2} + \&c.$$

the values of the differential coefficients being taken for

$$\omega = 0 \text{ (see 32)}$$

And

$$d\,z = p\,d\,\varrho = -\,p\,\varrho^2\,d\,\omega$$
$$d^2\,z = -\,2\,p\,\varrho\,d\,\varrho\,d\,\omega - \varrho^2\,d\,\omega\,d\,p$$

or, making

$$d\,p = q\,d\,\varrho$$
$$d^2\,z = -\,(2\,p + q\,\varrho)\,\varrho\,d\,\varrho\,d\,\omega = (2\,p + q\,\varrho)\,\varrho^3\,d\,\omega^2.$$

And if $p_{\prime}$ and $q_{\prime}$ be the values which p and q assume when $\omega = 0$, $\varrho = a$, we have for that case,

$$\frac{d^2\,z}{d\,\omega^2} = (2\,p_{\prime} + q_{\prime}\,a)\,a^3$$

$$Z = k - p_{\prime}\,a^2\omega + (2\,p + q_{\prime}\,a)\,a^3.\,\frac{\omega^2}{2} - \&c.$$

Also

$$\frac{1}{\varrho^2} = \left(\frac{1}{a} + \omega\right)^2 = \frac{1}{a^2} + \frac{2\,\omega}{a} + \omega^2.$$

Hence

$$2\,g\,(k - z) - h^2\left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)$$

becomes

$$2\,g\left(p_{\prime}\,a^2\,\omega - (2\,p_{\prime} + q_{\prime}\,a)\,a^3.\,\frac{\omega^2}{2} + \&c.\right) - h^2\left(\frac{2\,\omega}{a} + \omega^2\right).$$

But when a body moves in a circle of radius = a, we have

$$h^2 = g\,\varrho^3\,p = g\,a^3\,p_{\prime}$$

in this case. And when the body moves *nearly* in a circle, h^2 will have nearly this value. If we put

$$h^2 = (1 + \delta)\,g\,a^3\,p_{\prime}$$

we shall finally have to put

$$\delta = 0$$

$$\therefore dt = \frac{2 r d \theta}{\sqrt{2g . (\gamma - \beta)} . \sqrt{\{1 - \delta^2 \sin.^2 \theta\}}}$$

which is to be integrated from $z = \beta$, to $z = \alpha$; that is from

$$\theta = 0 \text{ to } \theta = \frac{\pi}{2}$$

this expanded in the same way as in 374 gives

$$t = \frac{2 r}{\sqrt{2 g (\gamma - \beta)}} \times \left\{ 1 + \left(\frac{\delta}{2}\right)^2 + \left(\frac{1.\ 3\ \delta^2}{2.\ 4}\right)^2 + \&c. \right\} \frac{\pi}{2}$$

which is the time of a whole oscillation from the least to the greatest distance.

Also

$$d \theta = \frac{h d t}{\varrho^2} = \frac{h d t}{2 r z - z^2}$$

and θ is hence known in terms of z.

383. *A body acted on by gravity moves on a surface of revolution whose axis is vertical: when its path is nearly circular, it is required to find the angle between the apsides of the path projected in the plane of* x, y.

In this case

$$\int Z d z = g z = Q$$

and if at an apse

$$\varrho = a, \ z = k$$

we have

$$(C - 2 g k) a^2 - h^2 = 0$$

$$\therefore C = \frac{h^2}{a^2} + 2 g k.$$

Hence (380)

$$d \theta = \frac{\sqrt{(1 + p^2)}\, h d \varrho}{\varrho \sqrt{\left\{2 g (k - z) \varrho^2 - \frac{h^2 (a^2 - \varrho^2)}{a^2}\right\}}}$$

$$= \frac{\sqrt{(1 + p^2)} \frac{h d \varrho}{\varrho^2}}{\sqrt{\left\{2 g (k - z) - h^2 \left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)\right\}}}$$

Let $\frac{1}{\varrho} = \frac{1}{a} + \omega$

$$\therefore - \frac{d \varrho}{\varrho^2} = d \omega$$

$$\therefore d \theta = - \frac{\sqrt{(1 + p^2)}\, h d \omega}{\sqrt{\left\{2 g (k - z) - h^2 \left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)\right\}}}$$

$$\therefore \frac{d\,\omega^2}{d\,\theta^2} = \frac{2\,g\,(k - z) - h^2\left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)}{h^2\,(1 + p^2)}.$$

It is requisite to express the right-hand side of this equation in terms of ω

Now since at an apse we have

$$\omega = 0,\ z = k, \text{ and } \varrho = a$$

we have generally

$$z = k + \frac{d\,z}{d\,\omega}\,\omega + \frac{d^2 z}{d\,\omega^2}\cdot\frac{\omega^2}{1.2} + \&c.$$

the values of the differential coefficients being taken for

$$\omega = 0 \text{ (see 32)}$$

And

$$d\,z = p\,d\,\varrho = -\,p\,\varrho^2\,d\,\omega$$
$$d^2 z = -\,2\,p\,\varrho\,d\,\varrho\,d\,\omega - \varrho^2\,d\,\omega\,d\,p$$

or, making

$$d\,p = q\,d\,\varrho$$
$$d^2 z = -\,(2\,p + q\,\varrho)\,\varrho\,d\,\varrho\,d\,\omega = (2\,p + q\,\varrho)\,\varrho^3\,d\,\omega^2.$$

And if $p_{\prime}$ and $q_{\prime}$ be the values which p and q assume when $\omega = 0$, $\varrho = a$, we have for that case,

$$\frac{d^2 z}{d\,\omega^2} = (2\,p_{\prime} + q_{\prime}\,a)\,a^3$$

$$Z = k - p_{\prime}\,a^2\omega + (2\,p + q_{\prime}\,a)\,a^3.\,\frac{\omega^2}{2} - \&c.$$

Also

$$\frac{1}{\varrho^2} = \left(\frac{1}{a} + \omega\right)^2 = \frac{1}{a^2} + \frac{2\,\omega}{a} + \omega^2.$$

Hence

$$2\,g\,(k - z) - h^2\left(\frac{1}{\varrho^2} - \frac{1}{a^2}\right)$$

becomes

$$2\,g\left(p_{\prime}\,a^2\,\omega - (2\,p_{\prime} + q_{\prime}\,a)\,a^3.\,\frac{\omega^2}{2} + \&c.\right) - h^2\left(\frac{2\,\omega}{a} + \omega^2\right).$$

But when a body moves in a circle of radius $=$ a, we have

$$h^2 = g\,\varrho^3\,p = g\,a^3\,p_{\prime}$$

in this case. And when the body moves *nearly* in a circle, h^2 will have nearly this value. If we put

$$h^2 = (1 + \delta)\,g\,a^3\,p_{\prime}$$

we shall finally have to put

$$\delta = 0$$

in order to get the ultimate angle when the orbit becomes indefinitely near a circle. Hence we may put

$$h^2 = g\,a^3\,p_{,}$$

and

$$2\,g\,(k - z) - h^2\left(\frac{1}{\varrho^3} - \frac{1}{a^2}\right)$$

becomes

$$-\{3\,g\,a^3\,p_{,} + g\,a^4\,q_{,}\}\,\omega^2 + \&c.$$

in which the higher powers of ω may be neglected in comparison of ω^2;

$$\therefore \frac{d\,\omega^2}{d\,\theta^2} = -\frac{g\,a^3\,(3\,p_{,} + q_{,}\,a)\,\omega^2}{h^2\,(1 + p^2)} = \frac{-(3\,p_{,} + q_{,}\,a)\,\omega^2}{p_{,}\,(1 + p^2)}$$

$$= -\frac{(3\,p_{,} + q_{,}\,a)\,\omega^2}{p_{,}\,(1 + p_{,}^2)}$$

again omitting powers above ω^2: for $p = p_{,} + A\,\omega + \&c.$

Differentiate and divide by $2\,d\,\omega$, and we have

$$\frac{d^2\,\omega}{d\,\theta^2} = -\frac{3\,p_{,} + q_{,}\,a}{p_{,}\,(1 + p_{,}^2)}\cdot\omega = -N\,\omega$$

suppose; of which the integral is taken so that

$$\theta = 0, \text{ when } \omega = 0$$

is

$$\omega = C\,\sin.\,\theta\,\sqrt{N}.$$

And ω passes from 0 to its greatest value, and consequently ϱ passes from the value a, to another maximum or minimum, while the arc $\theta\,\sqrt{N}$ passes from 0 to π. Hence, for the angle A between the apsides we have

$$A\,\sqrt{N} = \pi \text{ or } A = \frac{\pi}{\sqrt{N}}$$

where

$$N = \frac{3\,p_{,} + q_{,}\,a}{p_{,}\,(1 + p_{,}^2)}.$$

384. *Let the surface be a sphere and let the path described be nearly a circle; to find the horizontal angle between the apsides.*

Supposing the origin to be at the lowest point of the surface, we have

$$z = r - \sqrt{(r^2 - \varrho^2)}$$

$$p = \frac{d\,z}{d\,\varrho} = \frac{\varrho}{\sqrt{(r^2 - \varrho^2)}}$$

$$q = \frac{d\,p}{d\,\varrho} = \frac{r^2}{(r^2 - \varrho^2)^{\frac{3}{2}}}$$

$$\therefore p_{,} = \frac{a}{\sqrt{(r^2 - a^2)}}$$

$$q_{\prime} = \frac{r^2}{(r^2 - a^2)^{\frac{3}{2}}}$$

$$1 + p_{\prime}^{2} = \frac{r^2}{r^2 - a^2}$$

$$\therefore N = \frac{4r^2 - 3a^2}{r^2}.$$

Hence the angle between the apsides is

$$A = \frac{\pi r}{\sqrt{(4r^2 - 3a^2)}}.$$

The motion of a point on a spherical surface is manifestly the same as the motion of a simple pendulum or heavy body, suspended by an inextensible string from a fixed point; the body being considered as a point and the string without weight. If the pendulum begin to move in a vertical plane, it will go on oscillating in the same plane in the manner already considered. But if the pendulum have any lateral motion it will go on revolving about the lowest point, and generally alternately approaching to it, and receding from it. By a proper adjustment of the velocity and direction it may describe a circle (134); and if the velocity when it is moving parallel to the horizon be *nearly* equal to the velocity in a circle, it will describe a curve little differing from a circle. In this case we can find the angle between the greatest and least distances, by the formula just deduced.

Since

$$A = \frac{\pi r}{\sqrt{(4r^2 - 3a^2)}}$$

if $a = 0$, $A = \frac{\pi}{2}$, the apsides are 90° from each other, which also appears from observing that when the amplitude of the pendulum's revolution is very small, the force is nearly as the distance; and the body describes ellipses nearly; of which the lowest point is the center.

$$\text{If } a = r,$$

$$A = \pi = 180^\circ$$

this is when the pendulum string is horizontal; and requires an infinite velocity.

If $a = \frac{r}{2}$; so that the string is inclined 30° to the vertical;

$$A = \frac{2\pi}{\sqrt{13}} = 99^\circ\ 50'.$$

If $a^2 = \frac{r^2}{2}$; so that the string is inclined 45° to the vertical;

$$A = \pi \sqrt{\frac{2}{5}} = 113^\circ.\ 56'.$$

If $a^2 = \frac{3\,r^2}{4}$; so that the string is inclined 60° to the vertical;

$$A = \frac{2\,\pi}{\sqrt{7}} = 136^\circ \text{ nearly.}$$

385. *Let the surface be an inverted cone, with its axis vertical: to find the horizontal angle between the apsides when the orbit is nearly a circle.*

Let r be the radius of the circle and γ the angle which the slant side makes with the horizon. Then

$$z = \varrho \tan.\ \gamma$$
$$p = \tan.\ \gamma$$
$$q = 0$$
$$\therefore N = \frac{3 \tan.\ \gamma}{\tan.\ \gamma.\ \sec.^2 \gamma} = 3 \cos.^2 \gamma$$

and

$$A = \frac{\pi}{\cos.\ \gamma \sqrt{3}}.$$

$$\text{If } \gamma = 60^\circ$$
$$A = \frac{2\,\pi}{3} = 120^\circ.$$

386. *Let the surface be an inverted paraboloid whose parameter is* c.

$$\varrho^2 = c\,z$$
$$\therefore p = \frac{d\,z}{d\,\varrho} = \frac{2\,\varrho}{c}$$
$$q = \frac{2}{c}$$
$$\therefore N = \frac{\frac{6\,a}{c} + \frac{2\,a}{c}}{\frac{2\,a}{c}\left(1 + \frac{4\,a^2}{c^2}\right)} = \frac{4\,c^2}{c^2 + 4\,a^2}.$$

If $a = \frac{c}{2}$, or the body revolve at the extremity of the focal ordinate,

$$N = 2$$

and

$$A = \frac{\pi}{\sqrt{2}}.$$

387. *When a body moves on a conical surface, acted on by a force tending to the vertex; its motion in the surface will be the same, as if the surface were unwrapped, and made plane, the force remaining at the vertex.*

Measuring the radius-vector (ϱ) from the vertex, let the force be F, and the angle which the slant side makes with the base $= \gamma$; then

$$z = \varrho \tan. \gamma$$
$$p = \tan. \gamma$$
$$1 + p^2 = \sec.^2 \gamma$$

also

$$Q = \int (F \, d\varrho + Z \, dz) = \int F' \, d\varrho'.$$

Hence (380)

$$d\theta = \frac{\sec. \gamma \, h \, d\varrho}{\varrho \sqrt{\{(C - 2\int F' d\varrho') \varrho^2 - h^2\}}}$$

or putting

$$h' \cos. \gamma \text{ for } h$$
$$d\theta' \sec. \gamma \text{ for } d\theta$$

and

$$\varrho' \cos. \gamma \text{ for } \varrho$$

we have

$$d\theta' = \frac{h' \, d\varrho'}{\varrho' \sqrt{\{(C - 2\int F' d\varrho') \varrho'^2 - h'^2\}}}.$$

Now $d\theta'$ is the differential of the angle described *along the conical surface*, and it appears that the relation between θ' and ϱ' will be the same as in a plane, where a body is acted upon by a central force F. For in that case we have

$$d \left\{ \frac{h'^2 d\varrho'^2}{\varrho'^4 d\theta'^2} + \frac{h'^2}{\varrho'^2} \right\} = -2 F' d\varrho'$$

and integrating

$$\frac{h'^2 d\varrho'^2}{\varrho'^4 d\theta'^2} + \frac{h'^2}{\varrho'^2} = C - 2\int F' d\varrho'$$

which agrees with the equation just found.

388. *When a body moves on a surface of revolution, to find the reaction* R.

Take the three original equations (380) and multiply them by $x \, dz$, $y \, dz$, $\varrho \, d\varrho$; and the two first become

$$\frac{x \, d^2x \, dz}{dt^2} = -\frac{F' x^2 dz}{\varrho} - R \frac{dz^2}{ds} \cdot \frac{x^2}{\varrho}$$

$$\frac{y \, d^2y \, dz}{dt^2} = -\frac{F' y^2 dz}{\varrho} - R \frac{dz^2}{ds} \cdot \frac{y^2}{\varrho}$$

add these, observing that

$$x^2 + y^2 = \varrho^2$$

and we have

$$\frac{(x\,d^2x + y\,d^2y)\,dz}{dt^2} = -F'\varrho\,dz - R\varrho\frac{dz^2}{ds}.$$

Also the third is

$$\frac{\varrho\,d\varrho\,d^2z}{dt^2} = -Z\varrho\,d\varrho + R\varrho\frac{d\varrho^2}{ds}.$$

Subtract this, observing that $dz^2 + d\varrho^2 = ds^2$, and we have

$$\frac{(x\,d^2x + y\,d^2y)\,dz - \varrho\,d\varrho\,d^2z}{dt^2} =$$

$$\varrho(Z\,d\varrho - F'\,dz) - R\varrho\,ds.$$

But

$$x^2 + y^2 = \varrho^2$$

$$x\,dx + y\,dy = \varrho\,d\varrho$$

$$x\,d^2x + y\,d^2y + dx^2 + dy^2 = \varrho\,d^2\varrho + d\varrho^2.$$

Hence

$$\frac{(d\varrho^2 - dx^2 - dy^2)\,dz}{dt^2} + \frac{\varrho\,dz\,d^2\varrho - \varrho\,d\varrho\,d^2z}{dt^2} =$$

$$\varrho(Z\,d\varrho - F'\,dz) - R\varrho\,ds$$

and

$$d\varrho^2 = ds^2 - dz^2.$$

Hence

$$R = \frac{Z\,d\varrho - F'\,dz}{ds} + \frac{d\varrho\,d^2z - dz\,d^2\varrho}{dt^2\,ds}$$

$$+ \frac{(dx^2 + dy^2 + dz^2 - ds^2)\,dz}{\varrho\,dt^2\,ds}.$$

Now if r be the radius of curvature, we have (74)

$$r = \frac{ds^3}{d\varrho\,d^2z - dz\,d^2\varrho}$$

and

$$dx^2 + dy^2 + dz^2 = d\sigma^2$$

σ being the arc described.

Hence

$$R = \frac{Z\,d\varrho - F'\,dz}{ds} + \frac{ds^2}{r\,dt^2}$$

$$+ \frac{d\sigma^2 - ds^2}{\varrho\,dt^2}\cdot\frac{dz}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here it is manifest that

$$\frac{ds^2}{dt^2}$$

is the square of the velocity resolved into the generating curve, and that

$$\frac{d\sigma^2 - d s^2}{d t^2}$$

is the square of the velocity resolved perpendicular to ϱ. The two last terms which involve these quantities, form that part of the resistance which is due to the centrifugal force; the first term is that which arises from the resolved part of the forces.

From this expression we know the value of R; for we have, as before

$$\frac{d\sigma^2}{d t^2} = C - 2 \int (F' d\varrho + Z d z).$$

Also

$$\frac{d\sigma^2 - d s^2}{d t^2} = \frac{\varrho^2 d \theta^2}{d t^2} = \frac{h^2}{\varrho^2}.$$

Hence

$$\frac{d s^2}{d t^2} = C - 2 \int (F' d\varrho + Z d z)$$

$$= \frac{h^2}{\varrho^2}.$$

389. *To find the tension of a pendulum moving in a spherical surface.*

$$C - 2 \int (F' d\varrho + Z d z) = 2 g (k - z)$$

$$\varrho = \sqrt{(2 r z - z^2)}$$

$$\frac{d\varrho}{d z} = \frac{r - z}{\sqrt{(2 r z - z^2)}}$$

$$\frac{d s}{d\varrho} = \frac{r}{r - z}$$

$$\frac{d s}{d z} = \frac{r}{\sqrt{(2 r z - z^2)}} = \frac{r}{\varrho}.$$

Hence

$$R = \frac{g (r - z)}{r} + \frac{2 g (k - z) - \frac{h^2}{\varrho^2}}{r} + \frac{h^2}{\varrho^3} \cdot \frac{\varrho}{r}$$

$$= \frac{g (r + 2 k - 3 z)}{r}$$

and hence it is the same as that of the pendulum oscillating in a vertical plane with the same velocity at the same distances.

390. *To find the Velocity, Reaction, and Motion of a body upon any surface whatever.*

Let R be the reaction of the surface, which is in the direction of a normal to it at each point. Also let ι, ι', ι'' be the angles which this normal

makes with the axes of x, y, z respectively; we shall then have, considering the resolved parts of R among the forces which act on the point

$$\left.\begin{aligned}\frac{d^2 x}{d t^2} &= X + R \cos. \varepsilon \\ \frac{d^2 y}{d t^2} &= Y + R. \cos. \varepsilon' \\ \frac{d^2 z}{d t^2} &= Z + R. \cos. \varepsilon''\end{aligned}\right\}$$

Now the nature of the surface is expressed by an equation between x, y, z: and if we suppose that we have deduced from this equation

$$d z = p d x + q d y$$

$$\text{where } p = \frac{d z}{d x} \text{ and } q = \frac{d z}{d y},$$

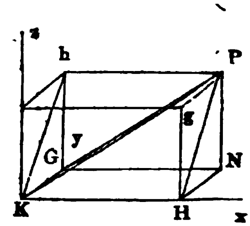
p and q being taken on the supposition of y and x being constants respectively; we have for the equations to the normal of the points whose coordinates are

$$x, y, z$$

$$\left.\begin{aligned}x' - x + p(z' - z) = 0 \\ y' - y + q(z' - z) = 0\end{aligned}\right\}$$

x′, y′, z′ being coordinates to any point in the normal (see Lacroix, No. 143.)

Hence it appears that if P K be the normal, P G, P H its projections on planes parallel to x z, y z respectively.



The equation of P G is

$$x' - x + p.(z' - z) = 0,$$

and hence

$$G N + p P N = 0$$

and

$$G N = -p . P N.$$

Similarly the equation of P H is

$$y' - y + q(z' - z) = 0$$

whence

$$H N + q . P N = 0$$

$$H N = -q . P N.$$

And hence,

$$\cos. \varepsilon = \cos. K P h = \frac{P h}{P K}$$

$$= \frac{G N}{\sqrt{(P N^2 + N G^2 + H N^2)}}$$

$$= -\frac{p}{\sqrt{(1 + p^2 + q^2)}}$$

$$\cos. \iota' = \cos. K P g = \frac{P g}{P K}$$

$$= -\frac{H N}{\sqrt{(P N^2 + N G^2 + H N^2)}}$$

$$= -\frac{q}{\sqrt{(1 + p^2 + q^2)}}.$$

Whence, since

$$\cos.^2 \iota + \cos.^2 \iota' + \cos.^2 \iota'' = 1$$

$$\cos.^2 \iota'' = \sqrt{(1 - \cos.^2 \iota - \cos.^2 \iota')}$$

$$= \frac{1}{\sqrt{(1 + p^2 + q^2)}}.$$

Substituting these values; multiplying by $d x$, $d y$, $d z$ respectively, in the three equations; and observing that

$$d z - p d x - q d y = 0$$

we have

$$\frac{d x d^2 x + d y d^2 y + d z d^2 z}{d t^2} = X d x + Y d y + Z d z$$

and integrating

$$\frac{d x^2 + d y^2 + d z^2}{d t^2} = 2 \int (X d x + Y d y + Z d z)$$

and if this can be integrated, we have the velocity.

If we take the three original equations, and multiply them respectively by $-p$, $-q$, and 1, and then add, we obtain

$$-p \frac{d^2 x}{d t^2} - q \cdot \frac{d^2 y}{d t^2} + \frac{d^2 z}{d t^2} =$$

$$-p X - q Y + Z + R \sqrt{(1 + p^2 + q^2)}.$$

But

$$d z = p d x + q d y.$$

Hence

$$\frac{d^2 z}{d t^2} = p \frac{d^2 x}{d t^2} + q \frac{d^2 y}{d t^2} + \frac{d p d x + d q d y}{d t^2}.$$

Substituting this on the first side of the above equation, and taking the value of R, we find

$$R = \frac{p X + q Y - Z}{\sqrt{(1 + p^2 + q^2)}} + \frac{d p d x + d q d y}{d t^2 \sqrt{(1 + p^2 + q^2)}}$$

If in the three original equations we eliminate R, we find two second differential equations, involving the known forces

X, Y, Z

and p, q, which are also known when the surface is known, combining with these the equation to the surface, by which z is known in terms of x, y, we have equations from which we can find the relation between the time and the three coordinates.

391. *To find the path which a body will describe upon a given surface, when acted upon by no force.*

In this case we must make

$$X, Y, Z \text{ each} = 0.$$

Then, if we multiply the three equations of the last art. respectively by

$$-(q\,d\,z + d\,y),\ p\,d\,z + d\,x,\ q\,d\,x - p\,d\,y$$

and add them, we find,

$$-(q\,d\,z + d\,y)\,d^2x + (p\,d\,z + d\,x)\,d^2y + (q\,d\,x - p\,d\,y)\,d^2z$$

$$= R\,d\,t^2 \left\{\begin{array}{l} -(q\,d\,z + d\,y)\cos.\iota \\ +(p\,d\,z + d\,x)\cos.\iota' \\ +(q\,d\,x - p\,d\,y)\cos.\iota'' \end{array}\right\}$$

or putting for cos. ι, cos. ι', cos. ι'' their values

$$= \frac{R\,d\,t^2}{\sqrt{(1+p^2+q^2)}}\{p(q\,d\,z+d\,y) - q\,(p\,d\,z+d\,x) + q\,d\,x - p\,d\,y\} = 0.$$

Hence, for the curve described in this case, we have

$$(p\,d\,z+d\,x)\,d^2y = (p\,d\,y - q\,d\,x)\,d^2z + (q\,d\,z + d\,y)\,d^2x.$$

This equation expresses a relation between x, y, z, without any regard to the time. Hence, we may suppose x the independent variable, and $d^2x = 0$; whence we have

$$(p\,d\,z + d\,x)\,d^2y = (p\,d\,y - q\,d\,x)\,d^2z.$$

This equation, combined with

$$d\,z = p\,d\,x + q\,d\,y,$$

gives the curve described, where the body is left to itself, and moves along the surface.

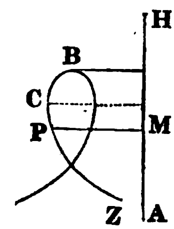
The curve thus described is the shortest line which can be drawn from one of its points to another, upon the surface.

The velocity is constant as appears from the equation

$$v^2 = 2\int(X\,d\,x + Y\,d\,y + Z\,d\,z).$$

By methods somewhat similar we might determine the motion of a point upon a given curve of double curvature, or such as lies not in one plane when acted upon by given forces.

392. *To find the curve of equal pressure, or that on which a body descending by the force of gravity, presses equally at all points.*



Let A M be the vertical abscissa = x, M P the horizontal ordinate = y; the arc of the curve s, the time t, and the radius of curvature at P = r, r being positive when the curve is concave to the axis; then R being the reaction at P, we have by what has preceded.

$$R = \frac{g\,d\,y}{d\,s} + \frac{d\,s^2}{r\,d\,t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But if H M be the height due to the velocity at P, A H = h, we have

$$\frac{d\,s^2}{d\,t^2} = 2\,g\,(h - x).$$

Also, if we suppose d s constant, we have (74)

$$r = -\frac{d\,s\,d\,x}{d^2\,y}$$

and if the constant value of R be k, equation (1) becomes

$$k = \frac{g\,d\,y}{d\,s} - \frac{2\,g\,(h - x)\,d^2\,y}{d\,s\,d\,x}$$

$$\therefore -\frac{k}{g} \cdot \frac{d\,x}{2\sqrt{(h - x)}} = \sqrt{(h - x)} \cdot \frac{d^2 y}{d\,s} - \frac{d\,y}{d\,s} \cdot \frac{d\,x}{2\sqrt{(h - x)}}$$

The right-hand side is obviously the differential of

$$\sqrt{(h - x)}\,\frac{d\,y}{d\,s};$$

hence, integrating

$$\frac{k}{g} \cdot \sqrt{(h - x)} = \sqrt{(h - x)} \cdot \frac{d\,y}{d\,s} + C,$$

$$\frac{d\,y}{d\,s} = \frac{k}{g} - \frac{C}{\sqrt{(h - x)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If C = 0, the curve becomes a straight line inclined to the horizon, which obviously answers the condition. The sine of inclination is $\frac{k}{g}$.

In other cases the curve is found by equation (2), putting

$$\sqrt{(d\,x^2 + d\,y^2)} \text{ for } d\,s$$

and integrating.

If we differentiate equation (2), d s being constant, we have

$$\frac{d^2\,y}{d\,s} = -\frac{C\,d\,x}{2\,(h - x)^{\frac{3}{2}}}$$

$$r = -\frac{d\,s\,d\,x}{d^2 y} = \frac{2\,(h - x)^{\frac{3}{2}}}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And if C be positive, r is positive, and the curve is concave to the axis.

We have the curve parallel to the axis, as at C, when $\frac{dy}{ds} = 0$, that is, when $\frac{k}{g} = \frac{C}{\sqrt{(h-x)}}$; when

$$x = h - \frac{C^2 g^2}{k^2}.$$

When x increases beyond this, the curve approaches the axis, and $\frac{dy}{dx}$ is negative; it can never become < -1; hence B the limit of x is found by making

$$\frac{k}{g} - \frac{C}{\sqrt{(h-x)}} = -1;$$

or

$$x = h - \frac{C^2 g^2}{(k+g)^2}.$$

If k be $< g$, as the curve descends towards Z, it approximates perpetually to the inclination, the sine of which is $\frac{k}{g}$.

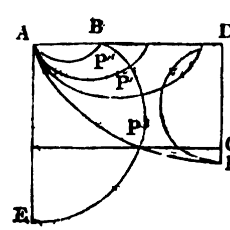
If k be $> g$ there will be a point at which the curve becomes horizontal.

C is known from (2), (3), if we knew the pressure or the radius of curvature at a given point.

If C be negative, the curve is convex to the axis. In this case the part of the pressure arising from centrifugal force diminishes the part arising from gravity, and k must be less than g.

393. *To find the curve which cuts a given assemblage of curves, so as to make them Synchronous, or descriptible by the force of gravity in the same time.*

Let A P, A P′, A P″, &c. be curves of the same kind, referred to a common base A D, and differing only in their parameters, (or the constants in their equations, such as the radius of a circle, the axes of an ellipse, &c.)



Let the vertical A M = x, M P (horizontal) = y; y and x being connected by an equation involving a. The time down A P is

$$\int \frac{dx}{\sqrt{(2gx)}}$$

the integral being taken between

$$x = 0 \text{ and } x = AM;$$

and this must be the same for all curves, whatever (a) may be.

Hence, we may put

$$\int \frac{d\,s}{\sqrt{(2\,g\,x)}} = k \;\; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

k being a constant quantity, and in differentiating, we must suppose (a) variable as well as x and s.

Let

$$d\,s = p\,d\,x$$

p being a function of x, and a which will be of 0 dimensions, because d x, and d s are quantities of the same dimensions. Hence

$$\int \frac{p\,d\,x}{\sqrt{(2\,g\,x)}} = k$$

and differentiating

$$\frac{p\,d\,x}{\sqrt{(2\,g\,x)}} + q\,d\,a = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Now, since p is of 0 dimensions in x, and a, it is easily seen that

$$\int \frac{p\,d\,x}{\sqrt{(2\,g\,x)}}$$

is a function whose dimensions in x and a are $\frac{1}{2}$, because the dimensions of an expression are increased by 1 in integrating. Hence by a known property of homogeneous functions, we have

$$\frac{p\,x}{\sqrt{(2\,g\,x)}} + q\,a = \frac{1}{2}\,k;$$

$$\therefore q = \frac{k}{2a} - \frac{p\,\sqrt{x}}{a\,\sqrt{(2\,g)}}$$

substituting this in equation (2) it becomes

$$\frac{p\,d\,x}{\sqrt{(2\,g\,x)}} + \frac{k\,d\,a}{2\,a} - \frac{p\,d\,a\,\sqrt{x}}{a\,\sqrt{(2\,g)}} = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

in which, if we put for (a) its value in x and y, we have an equation to the curve P P′ P″.

If the given time (k) be that of falling down a vertical height (h), we have

$$k = \sqrt{\frac{2\,h}{g}},$$

and hence, equation (3) becomes

$$p\,(a\,d\,x - x\,d\,a) + d\,a\,\sqrt{(h\,x)} = 0 \; . \; . \; . \; . \quad (4)$$

Ex. *Let the curves* A P, A P′, A P″ *be all cycloids of which the bases coincide with* A D.

Let C D be the axis of any one of these cycloids and = 2 a, a being the radius of the generating circle. If C N = x′, we shall have as before

$$-\,d\,s = d\,x'\sqrt{\frac{2\,a}{x'}}$$

and since

$$x' = 2a - x$$

$$ds = dx\sqrt{\frac{2a}{2a-x}}.$$

Hence

$$p = \sqrt{\frac{2a}{2a-x}},$$

and equation (4) becomes

$$\frac{\sqrt{(2a)}(a\,dx - x\,da)}{\sqrt{(2a-x)}} + d\sqrt{(hx)} = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Let $\frac{x}{a} = u$

so that

$$a\,dx - x\,da = a^2 du$$

$$x = au;$$

and substituting

$$\frac{a^2 du \sqrt{2}}{\sqrt{(2-u)}} + da\sqrt{(hau)} = 0$$

$$\therefore \frac{du\sqrt{2}}{\sqrt{(2u-u^2)}} + \frac{da\sqrt{h}}{a^{\frac{3}{2}}} = 0$$

$$\therefore \sqrt{2} \times \text{vers.}^{-1} u - 2\sqrt{\frac{h}{a}} = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When a is infinite, the portion A P of the cycloid becomes a vertical line, and

$$x = h, \ \therefore u = 0, \ \therefore C = 0.$$

Hence

$$\frac{x}{a} = \text{vers.}\sqrt{\frac{2h}{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From this equation (a) should be eliminated by the equation to the cycloid, which is

$$y = a\,\text{vers.}^{-1}\frac{x}{a} - \sqrt{(2ax - x^2)} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and we should have the equation to the curve required.

Substituting in (8) from (7), we have

$$y = \sqrt{(2ah)} - \sqrt{(2ax - x^2)}$$

$$dy = \frac{da\sqrt{h}}{\sqrt{(2a)}} - \frac{x\,da + a\,dx - x\,dx}{\sqrt{(2ax - x^2)}}$$

and eliminating d a by (5)

$$\frac{dy}{dx} = -\frac{2a-x}{\sqrt{(2ax-x^2)}} = -\sqrt{\frac{2a-x}{x}}.$$

But differentiating (8) supposing (a) constant, we have in the cycloid

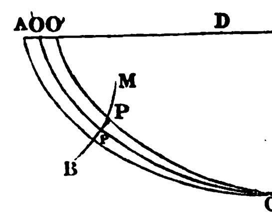
$$dy = \sqrt{\frac{x}{2a - x}}.$$

And hence (31) the curve P P′ P″ cuts the cycloids all at right angles, the subnormal of the former coinciding with the subtangent of the latter, each being

$$y\sqrt{\frac{2a - x}{x}}.$$

The curve P P′ P″ will meet A D in the point B, such that the given time is that of describing the whole cycloid A B. It will meet the vertical line in E, so that the body falls through A E in the given time.

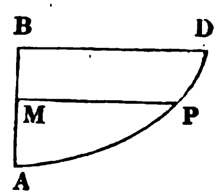
394. If instead of supposing all the cycloids to meet in the point A, we suppose them all to pass through any point C, their bases still being in the same line A D; a curve P P′ drawn so that the times down P C, P′ C, &c. are all equal, will cut all the cycloids at right angles. This may easily be demonstrated.



395. *To find Tautochronous curves or those down which to a given fixed point a body descending all distances shall move in the same time.*

(1) let the force be constant and act in parallel lines.

Let A the lowest point be the fixed point, D that from which the body falls, A B vertical, B D, M P horizontal. A M = x, A P = s, A B = h, and the constant force = g.



Then the velocity at P is

$$v = \sqrt{(2g \cdot \overline{h - x})}$$

and

$$dt = -\frac{ds}{v} = \frac{-ds}{\sqrt{2g}\sqrt{(h - x)}}$$

and the whole time of descent will be found by integrating this from x = h, to x = 0.

Now, since the time is to be the same, from whatever point D the body falls, that is whatever be h, the integral just mentioned, taken between the limits, must be independent of h. That is, if we take the integral so as to vanish when

$$x = 0$$

and then put h for x, h will disappear altogether from the result. This must manifestly arise from its being possible to put the result in a form

involving only $\frac{x}{h}$, as $\frac{x^2}{h^2}$, &c.; that is from its being of 0 dimensions in x and h.

Let

$$ds = p\,dx$$

where p depends only on the curve, and does not involve h. Then, we have

$$t = -\int \frac{p\,dx}{\sqrt{\{2g(h-x)\}}}$$
$$= -\frac{1}{\sqrt{(2g)}}\int\left\{\frac{p\,dx}{h^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{p\,x\,dx}{h^{\frac{3}{2}}} + \frac{1.3}{2.4}\,\frac{p\,x^2\,dx}{h^{\frac{5}{2}}} + \&c.\right\}$$

and from what has been said, it is evident, that each of the quantities

$$\int\frac{p\,dx}{h^{\frac{1}{2}}},\ \int\frac{p\,x\,dx}{h^{\frac{3}{2}}},\ \int\frac{p\,x^n\,dx}{h^{\frac{2n+1}{2}}}$$

must be of the form

$$\frac{c\,x^{\frac{2n+1}{2}}}{h^{\frac{2n+1}{2}}};$$

that is

$$\int p\,x^n\,dx \text{ must} = c\,x^{\frac{2n+1}{2}};$$

hence

$$p\,x^n\,dx = \frac{2n+1}{2}\,c\,x^{\frac{2n-1}{2}}\,dx;$$

$$p = \frac{2n+1}{2}\cdot\frac{c}{x^{\frac{1}{2}}};$$

or if

$$\frac{2n+1}{2}\,c = a^{\frac{1}{2}}$$

$$p = \sqrt{\frac{a}{x}}$$

and

$$ds = dx\sqrt{\frac{a}{x}}$$

which is a property of the cycloid.

Without expanding, the thing may thus be proved. If p be a function of m dimensions in x, $\frac{p}{\sqrt{(h-x)}}$ is of $m-\frac{1}{2}$ dimensions; and as the dimensions of an expression are increased by 1 in integrating

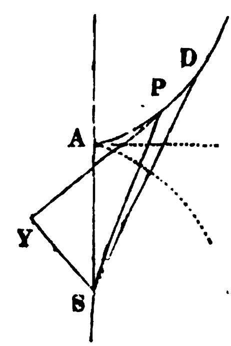
$$\int\frac{p\,dx}{\sqrt{(h-x)}}$$

is of $m + 1$ dimensions in x, and when h is put for x, of $m + \frac{1}{2}$ dimensions in h. But it ought to be independent of h or of 0 dimensions. Hence

$$m + \tfrac{1}{2} = 0$$
$$\therefore p = a^{\frac{1}{2}} x^{-\frac{1}{2}}$$

as before.

396. (2) *Let the force tend to a center and vary as any function of the distance. Required the Tautochronous Curve.*



Let S be the center of force, A the point to which the body must descend; D the point from which it descends. Let also

$$SA = e, SD = f, SP = \varrho, AP = s$$

P being any point whatever.

Now we have

$$v^2 = C - 2\int F d\varrho$$

or if

$$2 P d\varrho = \varphi(\varrho)$$
$$v^2 = \varphi(f) - \varphi(\varrho)$$

the velocity being 0 when $\varrho = f$.

Hence the time of describing D A is

$$t = \int -\frac{ds}{\sqrt{(\varphi f - \varphi \varrho)}}$$

taken from $\varrho = f$, to $\varrho = e$. And since the time must be the same whatever is D, the integral so taken must be independent of f.

Let

$$\varphi\varrho - \varphi e = z$$
$$\varphi f - \varphi e = h$$
$$ds = p\,dz$$

p depending on the nature of the curve, and not involving f. Then

$$t = -\int \frac{p\,dz}{\sqrt{(h-z)}}, \text{ from } z = h \text{ to } z = 0$$
$$= \int \frac{p\,dz}{\sqrt{(h-z)}}, \text{ from } z = 0 \text{ to } z = h.$$

And this must be independent of f, and therefore of φf, and of h. Hence, after taking the integral the result must be 0 when $z = 0$, and independent of h, when h is put for z. Therefore it must be of 0 dimensions in z and h. But if p be of n dimensions in z, or if

$$p = c z^n$$

$\frac{p}{\sqrt{(h-z)}}$ will be of $n - \frac{1}{2}$ dimensions,

and

$$\int \frac{p\,d\,z}{\sqrt{(h-z)}} \text{ of } n + \tfrac{1}{2} \text{ dimensions.}$$

Hence, $n + \frac{1}{2} = 0$, $n = -\frac{1}{2}$, and

$$p = \sqrt{\frac{C}{z}}$$

Therefore

$$d\,s = d\,z\sqrt{\frac{C}{z}} = \varphi'\varrho\,d\,\varrho\sqrt{\frac{C}{\varphi\varrho - \varphi e'}};$$

whence the curve is known.

If θ be the angle A S O, we have

$$d\,s^2 = d\,\varrho^2 + \varrho^2\,d\,\theta^2$$

and

$$d\,\theta^2 = \frac{d\,s^2 - d\,\varrho^2}{\varrho^2}$$

whence may be found a polar equation to the curve.

397. Ex. 1. Let the force vary as the distance, and be attractive. Then

$$F = \mu\varrho,\ \varphi\varrho = \mu\varrho^2;$$
$$z = \varphi\varrho - \varphi e = \mu(\varrho^2 - e^2);$$
$$d\,z = 2\mu\varrho\,d\,\varrho$$
$$d\,s = d\,z\sqrt{\frac{C}{z}} = 2\mu\varrho\,d\,\varrho\sqrt{\frac{C}{\mu(\varrho^2 - e^2)}}$$
$$= \varrho\,d\,\varrho\sqrt{\frac{4\,c\,\mu}{\varrho^2 - e^2}}$$

when $\varrho = e$, $\frac{d\,s}{d\,\varrho}$ is infinite or the curve is perpendicular to S A at A.

If S Y, perpendicular upon the tangent P Y, be called p, we have

$$\frac{p^2}{\varrho^2} = \frac{d\,s^2 - d\,\varrho^2}{d\,s^2}$$
$$= 1 - \frac{d\,\varrho^2}{d\,s^2}$$
$$= 1 - \frac{\varrho^2 - e^2}{4\,c\,\mu\,\varrho^2}$$
$$p^2 = \frac{e^2 - (1 - 4\,c\,\mu)\,\varrho^2}{4\,c\,\mu}.$$

If $e = 0$, or the body descend to the center, this gives the logarithmic spiral.

In other cases let

$$1 - 4\,c\,\mu = \frac{e^2}{a^2},$$

$$\therefore 4 c \mu = \frac{a^2 - e^2}{a^2}$$

and

$$p^2 = \frac{e^2 (a^2 - \rho^2)}{a^2 - e^2}$$

the equation to the Hypocycloid (370)

If $4 c \mu = 1$, the curve becomes a straight line, to which S A is perpendicular at A.

If $4 c \mu$ be $>$ 1 the curve will be concave to the center and go off to infinity.

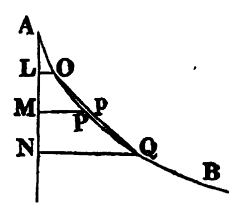
398. Ex. 2. Let the force vary inversely as the square of the distance. Then

$$F = \frac{\mu}{\rho^2};$$

and as before we shall find

$$p^2 = \rho^2 - \frac{\rho^5 (\rho - e)}{2 \mu c e}.$$

399. *A body being acted upon by a force in parallel lines, in its descent from one point to another, to find the Brachystochron, or the curve of quickest descent between them.*



Let A, B be the given points, and A O P Q B the required curve. Since the time down A O P Q B is less than down any other curve, if we take another as A O p Q B, which coincides with the former, except for the arc O P Q, we shall have

Time down A O : T. O P Q + T. Q B, less than
Time down A O + T. O p Q + T. Q B

and if the times down Q B be the same on the two suppositions, we shall have

T. O P Q less than the time down any other arc O p Q.

The times down Q B will be the same in the two cases if the velocity at Q be the same. But we know that the velocity acquired at Q is the same, whether the body descend down

A O P Q, or A O p Q.

Hence it appears that *if the time down* A O P Q B *be a minimum, the time down any portion* O P Q *is also a minimum.*

Let a vertical line of abscissas be taken in the direction of the force; and perpendicular ordinates, O L, P M, Q N be drawn, it being supposed that

$$\mathrm{L\,M} = \mathrm{M\,N}.$$

Then, if L M, M N be taken indefinitely small, we may consider them as representing the differential of x: On this supposition, O P, P Q, will represent the differentials of the curve, and the velocity may be supposed constant in O P, and in P Q. Let

$$\mathrm{A\,L} = x,\ \mathrm{L\,O} = y,\ \mathrm{O\,A} = s,$$

and let $d\,x$, $d\,y$, $d\,s$ be the differentials of the abscissa, ordinate, and curve at Q, and v the velocity there; and $d\,x'$, $d\,y'$, $d\,s'$, v' be the corresponding quantities at P. Hence the time of describing O P Q will be (46)

$$\frac{d\,s}{v} + \frac{d\,s'}{v'}$$

which is a minimum; and consequently its differential = 0. This differential is that which arises from supposing P to assume any position as p out of the curve O P Q; and as the differentials indicated by d arise from supposing P to vary its position along the curve O P Q, we shall use δ to indicate the differentiation, on hypothesis of passing from one curve to another, or the *variations* of the quantities to which it is prefixed.

We shall also suppose p to be in the line M P, so that $d\,x$ is not supposed to vary. These considerations being introduced, we may proceed thus,

$$\delta\left\{\frac{d\,s}{v} + \frac{d\,s'}{v'}\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And v, v′ are the same whether we take O P Q, or O p Q; for the velocity at p = velocity at P. Hence

$$\delta\,v = 0,\ \delta\,v' = 0$$

and

$$\frac{\delta\,d\,s}{v} + \frac{\delta\,d\,s'}{v'} = 0.$$

Now

$$d\,s^2 = d\,x^2 + d\,y^2$$

$$\therefore d\,s\,\delta\,d\,s = d\,y\,\delta\,d\,y,$$

(for $\delta\,d\,x = 0$).

Similarly

$$d\,s'\,\delta\,d\,s' = d\,y'\,\delta\,d\,y'.$$

Substituting the value of δ d s, δ d s' which these equations give, we have

$$\frac{\mathrm{d\,y}\,\delta\,\mathrm{d\,y}}{\mathrm{v\,d\,s}} + \frac{\mathrm{d\,y'}\,\delta\,\mathrm{d\,y'}}{\mathrm{v'\,d\,s'}} = 0.$$

And since the points O, Q, remain fixed during the variation of P's position, we have

$$\mathrm{d\,y} + \mathrm{d\,y'} = \text{const.}$$

$$\delta\,\mathrm{d\,y'} = -\delta\,\mathrm{d\,y}.$$

Substituting, and omitting δ d y,

$$\frac{\mathrm{d\,y}}{\mathrm{v\,d\,s}} - \frac{\mathrm{d\,y'}}{\mathrm{v'\,d\,s'}} = 0.$$

Or, since the two terms belong to the successive points O, P, their difference will be the differential indicated by d; hence,

$$\mathrm{d}\,.\,\frac{\mathrm{d\,y}}{\mathrm{v\,d\,s}} = 0$$

$$\therefore \frac{\mathrm{d\,y}}{\mathrm{v\,d\,s}} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Which is the property of the curve; and v being known in terms of x, we may determine its nature.

Let the force be gravity; then

$$\mathrm{v} = \sqrt{(2\,\mathrm{g\,x})};$$

$$\frac{\mathrm{d\,y}}{\mathrm{d\,s}\sqrt{(2\,\mathrm{g\,x})}} = \text{const.}$$

$$\frac{\mathrm{d\,y}}{\mathrm{d\,s}\sqrt{\mathrm{x}}} = \frac{1}{\sqrt{\mathrm{a}}}$$

a being a constant.

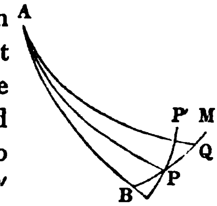
$$\therefore \frac{\mathrm{d\,y}}{\mathrm{d\,s}} = \sqrt{\frac{\mathrm{x}}{\mathrm{a}}}$$

which is a property of the cycloid, of which the axis is parallel to x, and of which the base passes through the point from which the body falls.

If the body fall from a given point to another given point, setting off with the velocity acquired down a given height; the curve of quickest descent is a cycloid, of which the base coincides with the horizontal line, from which the body acquires its velocity.

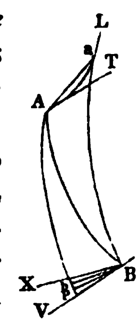
400. *If a body be acted on by gravity, the curve of its quickest descent from a given point to a given curve, cuts the latter at right angles.*

Let A be the given point, and B M the given curve; A B the curve of quickest descent cuts B M at right angles.



It is manifest the curve A B must be a cycloid, for otherwise a cycloid might be drawn from A to B, in which the descent would be shorter. If possible, let A Q be the cycloid of quickest descent, the angle A Q B being acute. Draw another cycloid A P, and let P P′ be the curve which cuts A P, A Q so as to make the arcs A P, A P′ *synchronous*. Then (394) P P′ is perpendicular to A Q, and therefore manifestly P′ is between A and Q, and the time down A P is less than the time down A Q; therefore, this latter is not the curve of quickest descent. Hence, if A Q be not perpendicular to B M, it is not the curve of quickest descent.

The cycloid which is perpendicular to B M may be the cycloid of *longest* descent from A to B M.



401. *If a body be acted on by gravity, and if* A B *be the curve of quickest descent from the curve* A L *to the point* B; A T, *the tangent of* A L *at* A, *is parallel to* B V, *a perpendicular to the curve* A B *at* B.

If B V be not parallel to A T, draw B X parallel to A T, and falling between B V and A. In the curve A L take a point *a* near to A. Let a B be the cycloid of quickest descent from the point a to the point B; and B b being taken equal and parallel to a A, let A b be a cycloid equal and similar to a B. Since A B V is a right angle, the curve B P, which cuts off A P synchronous to A B, has B V for a tangent. Also, ultimately A a coincides with A T, and therefore B b with B X. Hence B is between A and P. Hence, the time down A b is less than the time down A P, and therefore, than that down A B. And hence the time down a B (which is the same as that down A b) is less than that down A B. Hence, if B V be not parallel to A T, A B is not the line of quickest descent from A L to B.

402. *Supposing a body to be acted on by any forces whatever, to determine the Brachystochron.*

Making the same notations and suppositions as before, A L, L O, (see a preceding figure) being any rectangular coordinates; since, as before, the time down O P Q is a minimum, we have

$$\delta\left\{\frac{d\,s}{v}+\frac{d\,s'}{v'}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\delta d s}{v}+\frac{\delta d s'}{v'}-\frac{d s \delta v}{v^2}-\frac{d s' \delta v'}{v'^2}=0.$$

Now as before we also have

$$\delta d s=\frac{d y \delta d y}{d s}$$

supposing $\delta d x=0$, and

$$\delta d s'=\frac{d y'.\delta d y'}{d s'}=-\frac{d y'.\delta d y}{d s'}.$$

$$d v=0$$

for v is the velocity at O and does not vary by altering the curve.

$$v'=v+d v$$

$$d v'=\delta v+\delta d v=\delta d v.$$

Hence

$$\frac{d y \delta d y}{v d s}-\frac{d y' \delta d y}{v' d s'}-\frac{d s' \delta d v}{v'^2}=0.$$

Also

$$\frac{1}{v'}=\frac{1}{v+d v}=\frac{1}{v}-\frac{d v}{v^2};$$

for $d v^2$, &c. must be omitted. Substituting this in the second term of the above equation, we have

$$\frac{d y.\delta d y}{v d s}-\frac{d y' \delta d y}{v d s'}+\frac{d y' d v \delta d y}{v^2 d s'}-\frac{d s' \delta d v}{v'^2}=0$$

or

$$-\left(\frac{d y'}{d s'}-\frac{d y}{d s}\right).\frac{1}{v}+\frac{d y'.d v}{d s'.v^2}-\frac{d s'}{v'^2}.\frac{\delta d v}{\delta d y}=0$$

Now as before

$$\frac{d y'}{d s'}-\frac{d y}{d s}=d.\frac{d y}{d s}.$$

And in the other terms we may, since O, P, are indefinitely near, put

d s, d y, v for d s′, d y′, v′:

if we do this, and multiply by — v, we have

$$d.\frac{d y}{d s}-\frac{d y.d v}{d s.v}+\frac{d s}{v}.\frac{\delta d v}{\delta d y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which will give the nature of the curve.

If the forces which act on the body at O, be equivalent to X in the direction of x, and Y in the direction of y, we have (371)

$$v d v=X d x+Y d y$$

$$\therefore d v=\frac{X d x+Y d y}{v}$$

$$\therefore \delta d v=\frac{Y \delta d y}{v}$$

because $\delta v = 0$, $\delta d x = 0$; also X and Y are functions of A L, and L O, and therefore not affected by δ.

Substituting these values in the equation to the curve, we have

$$d \cdot \frac{dy}{ds} - \frac{dy}{ds} \cdot \frac{X dx + Y dy}{v^2} + \frac{ds}{v} \cdot \frac{Y}{v} = 0$$

or

$$d \cdot \frac{dy}{ds} - \frac{dx}{ds} \cdot \frac{X dy - Y dx}{v^2} = 0$$

which will give the nature of the curve.

If r be the radius of curvature, and d s constant, we have (from 74)

$$r = \frac{ds\, dx}{d^2 y}$$

r being positive when the curve is convex to A M;

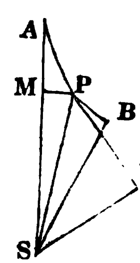
$$d \cdot \frac{dy}{ds} = \frac{dx}{r}$$

and hence

$$\frac{v^2}{r} = \frac{X dy - Y dx}{ds}.$$

The quantity $\frac{v^2}{r}$ is the centrifugal force (210), and therefore that part of the pressure which arises from it. And $\frac{X dy - Y dx}{ds}$ is the pressure which arises from resolving the forces perpendicular to the axis. Hence, it appears then in the Brachystochron for any given forces, the parts of the pressure which arise from the given forces and from the centrifugal force must be equal.

403. If we suppose the force to tend to a center S, which may be assumed to be in the line A M, and F to be the whole force; also if



$$S A = \alpha, S P = \varrho, S Y = p;$$

then we have

$\frac{X dy - Y dx}{ds}$ = force in P S resolved parallel to

$$Y S = F \times \frac{p}{\varrho}$$

and

$$v^2 = C - 2 \int g F d \varrho$$

$$\therefore \frac{C - 2 g \int F d\varrho}{r} = \frac{F p}{\varrho}$$

also

$$r = -\frac{\varrho\, d\varrho}{dp}$$

$$\therefore C - 2g\int F\,d\varrho = -\frac{F\,p\,d\varrho}{d\,p}$$

$$\therefore \frac{2\,dp}{p} = \frac{-2\,F\,d\varrho}{C - 2g\int F\,d\varrho}.$$

and integrating

$$p^2 = C'\,\{C - 2\,g\int F\,d\varrho\}$$

whence the relation of p and ϱ is known.

If the body begin to descend from A

$$C - 2\,g\int F\,d\varrho = 0$$

$$\text{when } \varrho = a.$$

404. Ex. 1. *Let the force vary directly as the distance.*

Here

$$F = \mu\,\varrho$$

$$C - 2\,g\int F\,d\varrho = v^2 = \mu\,g\,(a^2 - \varrho^2)$$

$$p^2 = C'\,\mu\,(a^2 - \varrho^2)$$

which agrees with the equation to the Hypocycloid (370).

405. Ex. 2. *Let the force vary inversely as the square of the distance*; then

$$F = \frac{\mu}{\varrho^2}$$

$$C - 2\,g\int F\,d\varrho = \frac{2\,g\,\mu}{\varrho} - \frac{2\,g\,\mu}{a}$$

$$p^2 = \frac{2\,\mu\,C'}{a}\,.\,\frac{a - \varrho}{\varrho} = c^2\,.\,\frac{a - \varrho}{\varrho}$$

by supposition.

$$\therefore \varrho^2 - p^2 = \frac{\varrho^3 + c^2\,\varrho - c^2\,a}{\varrho},$$

$$d\,\theta = \frac{p\,d\varrho}{\varrho\,\sqrt{(\varrho^2 - p^2)}}$$

$$= \frac{c\,\sqrt{(a - \varrho)}\,.\,d\varrho}{\varrho\,\sqrt{(\varrho^3 + c^2\,\varrho - c^2\,a)}}$$

$$= \frac{c\,d\varrho}{\varrho\sqrt{\left\{\frac{\varrho^3}{a - \varrho} - c^2\right\}}}$$

When $\varrho = a$, $d\,\theta = 0$; when

$$\varrho^3 + c^2\,\varrho - c^2\,a = 0$$

$d\,\theta$ is infinite, and the curve is perpendicular to the radius as at B. This equation has only one real root.

If we have $c = \frac{a}{2}$, $S\,B = \frac{a}{2}$

B being an apse.

$$\text{If } c = \frac{a}{10}, \ S B = \frac{a}{5}.$$

$$\text{If } c = \frac{a}{30}, \ S B = \frac{a}{10}.$$

$$\text{If } c = \frac{a}{n^3 + n}, \ S B = \frac{a}{n^2 + 1}.$$

406. *When a body moves on a given surface, to determine the Brachystochron.*

Let x, y, z be rectangular coordinates, x being vertical; and as before let d s, d s' be two successive elements of the curve; and let

$$d x, d y, d z,$$
$$d x', d y', d z'$$

be the corresponding elements of x, y, z; then since the minimum property will be true of the indefinitely small portion of the curve, we have as before, supposing v, v' the velocities,

$$\frac{d s}{v} + \frac{d s'}{v'} = \text{min.}$$

$$\therefore \delta . \left\{ \frac{d s}{v} + \frac{d s'}{v'} \right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The variations indicated by δ are those which arise, supposing d x, d x' to be equal and constant, and d y, d z, d y', d z' to vary

Now

$$d s^2 = d x^2 + d y^2 + d z^2$$
$$\therefore d s \delta d s = d y \delta d y + d z \delta d z.$$

Similarly

$$d s' \delta d s' = d y' \delta d y' + d z' \delta d z.$$

Also, the extremities of the arc

$$d s + d s'$$

being fixed, we have

$$d y + d y' = \text{const.}$$
$$\therefore \delta d y + \delta d y' = 0$$
$$d z + d z' = \text{const.}$$
$$\therefore \delta d z + \delta d z' = 0.$$

Hence

$$\left. \begin{aligned} \delta d s &= \frac{d y}{d s} \delta d y + \frac{d z}{d s} \delta d z \\ \delta d s' &= -\frac{d y'}{d s'} \delta d y - \frac{d z'}{d s'} \delta d z \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And the surface is defined by an equation between x, y, z, which we may call

$$L = 0.$$

Let this differentiated give

$$d z = p d x + q d y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, since d x, p, q are not affected by δ

$$\delta d z = q \delta d y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the sake of simplicity, we will suppose the body to be acted on only by a force in the direction of x, so that v, v' will depend on x alone, and will not be affected by the variation of d y, d z. Hence, we have by (1)

$$\frac{\delta d s}{v} + \frac{\delta d s'}{v'} = 0$$

which, by substituting from (2) becomes

$$\left\{\frac{\delta y'}{v' d s'} - \frac{d y}{v d s}\right\} \delta d y + \left\{\frac{d z'}{v' d s'} - \frac{d z}{v d s}\right\} \delta d z = 0.$$

Therefore we shall have, as before

$$d . \frac{d y}{v d s} \delta d y + d . \frac{d z}{v d s} \delta d z = 0;$$

and by equation (4), this becomes

$$d . \frac{d y}{v d s} + q d . \frac{d z}{v d s} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence the equation to the curve is known.

If we suppose the body not to be acted on by any force, v will be constant, and the path described will manifestly be the shortest line which can be drawn on the given surface, and will be determined by

$$d . \frac{d y}{d s} + q . d . \frac{d z}{d s} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

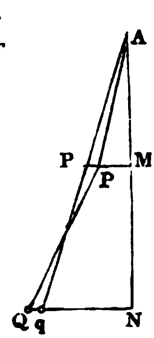
If we suppose d s to be constant, we have

$$d^2 y + q d^2 z = 0$$

which agrees with the equation there deduced for the path, when the body is acted on by no forces.

Hence, it appears that when a body moves along a surface undisturbed, it will describe the shortest line which can be drawn on that surface, between any points of its path.

407. *Let* P *and* Q *be two bodies, of which the first hangs from a fixed point and the second from the first by means of inextensible strings* A P, P Q; *it is required to determine the small oscillations.*



Let

$$A M = x, M P = y,$$
$$A N = x', N Q = y'$$
$$A P = a, P Q = a'$$
$$\text{mass of } P = \mu, \text{ of } Q = \mu'$$
$$\text{tension of } A P = p, \text{ of } P Q = p'.$$

Then resolving the forces p, p′, we have

$$\left.\begin{aligned}\frac{d^2 y}{d t^2} &= \frac{p' g}{\mu} \cdot \frac{y' - y}{a'} - \frac{p g}{\mu} \cdot \frac{y}{a} \\ \frac{d^2 y'}{d t^2} &= -\frac{p' g}{\mu'} \cdot \frac{y' - y}{a'}\end{aligned}\right\} \quad \dots \dots \quad (1)$$

By combining these with the equations in x, x′ and with the two

$$x^2 + y^2 = a^2,$$

$$(x' - x)^2 + (y' - y)^2 = a'^2;$$

we should, by eliminating p, p′ find the motion. But when the oscillations are small, we may approximate in a more simple manner.

Let β, β' be the initial values of y, y′. Then manifestly, p, p′ will depend on the initial position of the bodies, and on their position at the time t: and hence we may suppose

$$p = M + P\beta + Q\beta' + R y + S y' + \&c.$$

and similarly for p′.

Now, in the equations of motion above, p, p′ are multiplied by y, y′ — y which, since the oscillations are very small are also very small quantities, (viz. of the order β). Hence their products with β will be of the order β^2, and may be neglected, and we may suppose p reduced to its first term M.

M is the tension of A P, when β, β' &c. are all = 0. Hence it is the tension when P, Q, hang at rest from A, and consequently

$$M = \mu + \mu'.$$

Similarly, the first term of p′, which may be put for it is m′. Substituting these values and dividing by g, equations (1) become

$$\left.\begin{aligned}\frac{d^2 y}{g\, d t^2} &= -\left(\frac{\mu'}{\mu a'} + \frac{\mu + \mu'}{\mu a}\right) y + \frac{\mu'}{\mu a'}\, y' \\ \frac{d^2 y'}{g\, d t^2} &= \frac{y}{a'} - \frac{y'}{a'}\end{aligned}\right\} \quad \dots \quad (2$$

Multiply the second of these equations by λ and add it to the first, and we have

$$\frac{d^2 y + \lambda d^2 y'}{g\, d t^2} = -\left(\frac{\mu'}{\mu a'} + \frac{\mu + \mu'}{\mu a} - \frac{\lambda}{a'}\right) y - \left(\frac{\lambda}{a'} \cdot - \frac{\mu}{\mu a'}\right) y'$$

and manifestly this can be solved if the second member can be put in the form

$$-k \cdot (y + \lambda y')$$

that is, if

$$k = \frac{\mu'}{\mu a'} + \frac{\mu + \mu'}{\mu a} - \frac{\lambda}{a'}$$

$$k\lambda = \frac{\lambda}{a'} - \frac{\mu'}{\mu a'}$$

or

$$\left.\begin{aligned} a' k &= \frac{\mu'}{\mu} + \frac{a'}{a} + \frac{\mu' a'}{\mu a} - \lambda \\ -\frac{\mu'}{\mu} &= (a' k - 1)\lambda \end{aligned}\right\} \quad \ldots\ldots\ldots (3)$$

Eliminating λ we have

$$(a'k - 1)\, a'k - \frac{\mu'}{\mu} = (a'k - 1)\left(\frac{\mu'}{\mu} + \frac{a'}{a} + \frac{\mu' a'}{\mu a}\right)$$

Hence

$$(a'k)^2 - \left(1 + \frac{\mu'}{\mu}\right)\left(1 + \frac{a'}{a}\right) a'k = -\frac{a'}{a} - \frac{\mu' a'}{\mu a} \quad \ldots\ldots (4)$$

From this equation we obtain two values of k. Let these be denoted by

$${}^1k,\ {}^2k$$

and let the corresponding values of λ, be

$${}^1\lambda,\ {}^2\lambda.$$

Then, we have these equations.

$$\frac{d^2 y + {}^1\lambda\, d^2 y'}{g\, d t^2} = -{}^1k\,(y + {}^1\lambda\, y')$$

$$\frac{d^2 y + {}^2\lambda\, d^2 y'}{g\, d t^2} = -{}^2k\,(y + {}^2\lambda\, y')$$

and it is easily seen that the integrals of these equations are

$$y + {}^1\lambda\, y' = {}^1C \cos. t \surd({}^1k\, g) + {}^1D \sin. t \surd({}^1k\, g)$$
$$y + {}^2\lambda\, y' = {}^2C \cos. t \surd({}^2k\, g) + {}^2D \sin. t \surd({}^2k\, g)$$

${}^1C$, ${}^1D$, ${}^2C$, ${}^2D$ being arbitrary constants. But we may suppose

$${}^1C = {}^1E \cos. {}^1e$$
$${}^1D = {}^1E \sin. {}^1e$$
$${}^2C = {}^2E \cos. {}^2e$$
$$D^2 = {}^2E \sin. {}^2e$$

By introducing these values we find

$$\left.\begin{aligned} y + {}^1\lambda\, y' &= {}^1E \cos. \{t \surd({}^1k\, g) + {}^1e\} \\ y + {}^2\lambda\, y' &= {}^2E \cos. \{t \surd({}^2k\, g) + {}^2e\} \end{aligned}\right\} \quad \ldots\ldots (5)$$

From these we easily find

$$\left.\begin{aligned} y &= \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} \cos. \{t \surd({}^1k\, g) + {}^1e\} + \frac{{}^1\lambda\, {}^2E}{{}^2\lambda - {}^1\lambda} \cos. \{t \surd({}^2k\, g) + {}^2e\} \\ y' &= \frac{{}^1E}{{}^1\lambda - {}^2\lambda} \cos. \{t \surd({}^1k\, g) + {}^1e\} + \frac{{}^2E}{{}^1\lambda - {}^2\lambda} \cos. \{t \surd({}^2k\, g) + {}^2e\} \end{aligned}\right\} \ldots (6)$$

The arbitrary quantities ${}^1E$, ${}^1e$, &c. depend on the initial position and

velocity of the points. If the velocities of P, Q = 0, when t = 0, we shall have

$${}^1E,\ {}^2e, \text{ each } = 0$$

as appears by taking the Differentials of y, y′.

If either of the two ${}^1E$, ${}^2E$ be = 0, we shall have (supposing the latter case and omitting ${}^1e$)

$$y = \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} \cos. t \sqrt{({}^1k\, g)}$$

$$y' = \frac{{}^1E}{{}^1\lambda - {}^2\lambda} \cos. t \sqrt{({}^1k\, g)}.$$

Hence it appears that the oscillations in this case are *symmetrical:* that is, the bodies P, Q come to the vertical line at the same time, have similar and equal motions on the two sides of it, and reach their greatest distances from it at the same time. It is easy to see that in this case, the motion has the same law of time and velocity as in a cycloidal pendulum; and the time of an oscillation, in this case, extends from when t = 0 to when $t \sqrt{({}^1k\, g)} = \pi$. Also if β, β' be the greatest horizontal deviation of P, Q, we shall have

$$y = \beta . \cos. t \sqrt{({}^1k\, g)}$$
$$y' = \beta' . \cos. t \sqrt{({}^1k\, g)}.$$

In order to find the original relation of β, β', (the oscillations will be symmetrical if the forces which urge P, Q to the vertical be as P M, Q N, as is easily seen. Hence the conditions for symmetrical oscillation might be determined by finding the position of P, Q that this might originally be the relation of the forces) that the oscillations may be of this kind, the original velocities being 0, we must have by equation (5) since ${}^2E = 0$.

$$\beta + {}^2\lambda\, \beta' = 0.$$

Similarly, if we had

$$\beta + {}^1\lambda\, \beta' = 0$$

we should have ${}^1E = 0$, and the oscillations would be symmetrical, and would employ a time

$$\frac{\pi}{\sqrt{({}^2k\, g)}}.$$

When neither of these relations obtains, the oscillations may be considered as compounded of two in the following manner: Suppose that we put

$$y = H \cos. t \sqrt{({}^1k\, g)} + K \cos. t \sqrt{({}^2k\, g)} \quad . \quad . \quad . \quad (7)$$

omitting ${}^1e$, ${}^2e$, and altering the constants in equation (6); and suppose that we take

$$M\, p = H . \cos. t \sqrt{({}^1k\, g)};$$

Then p will oscillate about M according to the law of a cycloidal pendulum (neglecting the vertical motion). Also

$$p\,P \text{ will} = K.\cos. t \sqrt{({}^2k\,g)}.$$

Hence, P oscillates about p according to a similar law, while p oscillates about M. And in the same way, we may have a point q so moved, that Q shall oscillate about q in a time

$$\frac{\pi}{\sqrt{({}^2k\,g)}}$$

while q oscillates about N in a time

$$\frac{\pi}{\sqrt{({}^1k\,g)}}$$

And hence, the motion of the pendulum A P Q is compounded of the motion A p q oscillating symmetrically about a vertical line, and of A P Q oscillating symmetrically about A p q, as if that were a fixed vertical line.

When a pendulum oscillates in this manner it will never return exactly to its original position if $\sqrt{{}^1k}$, $\sqrt{{}^2k}$ are incommensurable.

If $\sqrt{{}^1k}$, $\sqrt{{}^2k}$ are commensurable so that we have

$$m\sqrt{{}^1k} = n\sqrt{{}^2k}$$

m and n being whole numbers, the pendulum will at certain intervals, return to its original position. For let

$$t\sqrt{({}^1k\,g)} = 2\,n\,\pi$$

then

$$t\sqrt{({}^2k\,g)} = 2\,m\,\pi$$

and by (7)

$$y = H\cos. 2\,n\,\pi + K.\cos. 2\,m\,\pi$$
$$= H + K,$$

which is the same as when

$$t = 0.$$

And similarly, after an interval such that

$$t\sqrt{({}^1k\,g)} = 4\,n\,\pi,\ 6\,n\,\pi,\ \&c.$$

the pendulum will return to its original position, having described in the intermediate times, similar cycles of oscillations.

408. Ex. *Let* $\mu' = \mu$

$$a' = a$$

to determine the oscillations.

Here equation (4) becomes

$$a^2 k^2 - 4\,a\,k = -2$$

and

$$a\,k = 2 \pm \sqrt{2}.$$

Also, by equation (3)

$$a k = 3 - \lambda$$

$$\therefore {}^1\lambda = 1 + \sqrt{2},\ {}^2\lambda = 1 - \sqrt{2}.$$

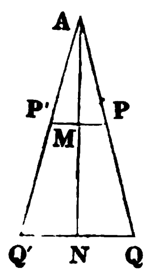
Hence, in order that the oscillations may be symmetrical, we must either have

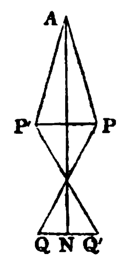
$$\beta + (1 + \sqrt{2})\,\beta' = 0, \text{ whence } \beta' = -(\sqrt{2} - 1)\,\beta$$

or

$$\beta - (\sqrt{2} - 1)\,\beta' = 0, \text{ whence } \beta' = (\sqrt{2} + 1)\,\beta.$$

The two arrangements indicated by these equations are thus represented.





The first corresponds to

$$\beta' = (\sqrt{2} + 1)\,\beta$$

or

$$Q N = (\sqrt{2} + 1)\, P M.$$

In this case, the pendulum will oscillate into the position A P′ Q′, similarly situated on the other side of the line; and the time of this complete oscillation will be

$$\frac{\pi}{\sqrt{\left\{\frac{g}{a}(2 - \sqrt{2})\right\}}} = \frac{\pi}{\sqrt{(2 - \sqrt{2})}}\sqrt{\frac{a}{g}}.$$

In the other case, corresponding to

$$\beta' = -(\sqrt{2} - 1)\,\beta$$

Q is on the other side of the vertical line, and

$$Q N = (\sqrt{2} - 1)\, P M.$$

The pendulum oscillates into the position A P′ Q′, the point O remaining always in the vertical line; and the time of an oscillation is

$$\frac{\pi}{\sqrt{(2 + \sqrt{2})}}\sqrt{\frac{a}{g}}.$$

The lengths of simple pendulums which would oscillate respectively in these times would be

$$\frac{a}{2 - \sqrt{2}} \text{ and } \frac{a}{2 + \sqrt{2}}$$

or

$$1.707\, a \text{ and } .293\, a.$$

If neither of these arrangements exist originally, let β, β' be the original values of y, y' when t is 0. Then making t = 0 in equation (5), we have

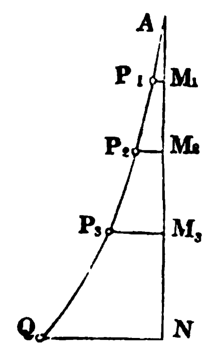
$$^1E = \beta + (\sqrt{2} + 1)\,\beta'$$

and

$$^2E = \beta - (\sqrt{2} - 1)\,\beta'.$$

And these being known, we have the motion by equation (6).

409. *Any number of material points* P_1, P_2, $P_3 \ldots Q$, *hang by means of a string without weight, from a point* A; *it is required to determine their small oscillations in a vertical plane.*



Let A N be a vertical abscissa, and $P_1 M_1$, $P_2 M_2$, &c. horizontal ordinates; so that

$$\begin{aligned} A\,M_1 &= x_1,\ A\,M_2 = x_2, \text{ \&c.} \\ P_1\,M_1 &= y_1,\ P_2\,M_2 = y_2, \text{ \&c.} \\ A\,P_1 &= a_1,\ P_1\,P_2 = a_2, \text{ \&c.} \\ \text{tension of } A\,P_1 &= p_1, \text{ of } P_1 P_2 = p_2, \text{ \&c.} \\ \text{mass of } P_1 &= \mu_1, \text{ of } P_2 = \mu_2, \text{ \&c.} \end{aligned}$$

Hence, we have three equations, by resolving the forces parallel to the horizon.

$$\left.\begin{aligned} \frac{d^2 y_1}{d t^2} &= -\frac{p_1 g}{\mu_1}\cdot\frac{y_1}{a_1} + \frac{p_2 g}{\mu_1}\cdot\frac{y_2 - y_1}{a_2} \\ \frac{d^2 y_2}{d t^2} &= -\frac{p_2 g}{\mu_2}\cdot\frac{y_2 - y_1}{a_2} + \frac{p_3 g}{\mu_2}\cdot\frac{y_3 - y_2}{a_3} \\ \frac{d^2 y_3}{d t^2} &= -\frac{p_3 g}{\mu_3}\cdot\frac{y_3 - y_2}{a_3} + \frac{p_4 g}{\mu_3}\cdot\frac{y_4 - y_3}{a_4} \\ &\text{- - - - - - - - - - - - -} \\ \frac{d^2 y_n}{d t^2} &= \frac{p_n g}{\mu_n}\cdot\frac{y_n - y_{n-1}}{a_n} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

And as in the last, it will appear that p_1, p_2, &c. may, for these small oscillations, be considered constant, and the same as in the state of rest. Hence if

$$\mu_1 + \mu_2 + \ldots\ldots \mu_n = M,$$

then

$$p_1 = M,\ p_2 = M - \mu_1,\ p_3 = M - \mu_1 - \mu_2, \text{ \&c.}$$

Also, dividing by g, and arranging, the above equations may be put in this form:

$$\left.\begin{aligned}
\frac{d^2 y_1}{g\,d\,t^2} &= -\left(\frac{p_1}{\mu_1 a_1} + \frac{p^2}{\mu_1 a_2}\right) y_1 + \frac{p_2 y_2}{\mu_1 a_2} \\
\frac{d^2 y_2}{g\,d\,t^2} &= \frac{p_2 y_1}{\mu_2 a_2} - \left(\frac{p_2}{\mu_2 a_2} + \frac{p_3}{\mu_2 a_3}\right) y_2 + \frac{p_3 y_3}{\mu_2 a_3} \\
\frac{d^2 y_3}{g\,d\,t^2} &= \frac{p_3 y_2}{\mu_3 a_3} - \left(\frac{p_3}{\mu_3 a_3} + \frac{p_4}{\mu_3 a_4}\right) y_3 + \frac{p_4 y_4}{\mu_3 a_4} \\
&- - - - - - - - - - - - - \\
\frac{d^2 y_n}{g\,d\,t^2} &= \frac{p_n y_{n-1}}{\mu_n a_n} - \frac{p_n y_n}{\mu_n a_n}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

The first and last of these equations become symmetrical with the rest if we observe that

$$y_o = 0$$

and

$$p_{n+1} = 0.$$

Now if we multiply these equations respectively by

$$1, \lambda, \lambda', \lambda'', \&c.$$

and add them, we have

$$\frac{d^2 y_1 + \lambda\, d^2 y_2 + \lambda'\, d^2 y_3 + \&c.}{g\,d\,t^2} =$$

$$\begin{aligned}
&\left\{-\frac{p_1}{\mu_1 a_1} - \frac{p_2}{\mu_1 a_2} + \frac{\lambda\, p_2}{\mu_2 a_2}\right\} y_1 \\
+ &\left\{\frac{p_2}{\mu_1 a_2} - \lambda\left(\frac{p_2}{\mu_2 a_2} + \frac{p_3}{\mu_2 a_3}\right) + \frac{\lambda'\, p_3}{\mu_3 a_3}\right\} y_2 \\
+ &\left\{\frac{\lambda\, p_3}{\mu_3 a_3} - \lambda'\left(\frac{p_3}{\mu_3 a_3} + \frac{p_4}{\mu_3 a_4}\right) + \frac{\lambda''\, p_4}{\mu_4 a_4}\right\} y_3 \\
&- - - - - - - - - - - - \\
+ &\left\{\frac{\lambda'^{(n-3)}\, p_n}{\mu_{n-1} a_n} - \frac{\lambda'^{(n-2)}\, p_n}{\mu_n a_n}\right\} y_n
\end{aligned}$$

and this will be integrable, if the right-hand side of the equation be reducible to this form

$$-k\,(y_1 + \lambda\, y_2 + \lambda'\, y_3 + \&c.).$$

That is, if

$$\left.\begin{aligned}
k &= \frac{p_1}{\mu_1 a_1} + \frac{p_2}{\mu_1 a_2} - \frac{\lambda\, p_2}{\mu_2 a_2} \\
k\,\lambda &= -\frac{p_2}{\mu_1 a_2} + \lambda\left(\frac{p_2}{\mu_2 a_2} + \frac{p_3}{\mu_2 a_3}\right) - \frac{\lambda'\, p_3}{\mu_3 a} \\
k\,\lambda' &= -\frac{\lambda\, p_3}{\mu_3 a_3} + \lambda'\left(\frac{p_3}{\mu_3 a_3} + \frac{p_4}{\mu_3 a_4}\right) + \frac{\lambda''\, p_4}{\mu_4 a_4} \\
&- - - - - - - - - - - - - \\
k\,\lambda'^{(n-2)} &= -\frac{\lambda'^{(n-3)}\, p_n}{\mu_{n-1} a_n} + \frac{\lambda'^{(n-2)}\, p_n}{\mu_n a_n}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

If we now eliminate

$$\lambda, \lambda', \lambda'', \&c.$$

from these n equations, it is easily seen that we shall have an equation of n dimensions in k.

Let

$$^1k, {}^2k, {}^3k \ldots\ldots {}^nk$$

be the n values of k; then for each of these there is a value of

$$\lambda', \lambda'', \lambda'''$$

easily deducible from equations (3), which we may represent by

$$^1\lambda, {}^1\lambda', {}^1\lambda'', \&c.$$
$$^2\lambda', {}^2\lambda'', {}^2\lambda''', \&c.$$

Hence we have these equations by taking corresponding values λ and k,

$$\frac{d^2 y_2 + {}^1\lambda\, d^2 y_2 + {}^1\lambda'\, d^2 y_3 + \&c.}{g\, d\, t^2} = -{}^1k\,(y_1 + {}^1\lambda\, y_2 + {}^1\lambda'\, y_3 + \&c.)$$

$$\frac{d^2 y_1 + {}^2\lambda\, d^2 y_2 + {}^2\lambda''\, d^2 y_3 + \&c.}{g\, d\, t^2} = -{}^2k\,(y_2 + {}^2\lambda\, y_2 + {}^2\lambda'\, y_3 + \&c.)$$

and so on, making n equations.

Integrating each of these equations we get, as in the last problem

$$\left.\begin{array}{l} y_1 + {}^1\lambda\, y_2 + {}^1\lambda'\, y_3 + \&c. = {}^1E \cos.\{t \sqrt{({}^1k\, g)} + {}^1e\} \\ y_1 + {}^2\lambda\, y_2 + {}^2\lambda'\, y_3 + \&c. = {}^2E \cos.\{t \sqrt{({}^2k\, g)} + {}^2e\} \end{array}\right\} \quad \ldots \quad (5)$$

^1E, ^2E, &c. ^1e, ^2e, &c. being arbitrary constants.

From these n simple equations, we can, without difficulty, obtain the n quantities y_1, y_2, &c. And it is manifest that the results will be of this form

$$\left.\begin{array}{l} y_1 = {}^1H_1 \cos.\{t \sqrt{({}^1k\, g)} + {}^1e\} + {}^2H_1 \cos.\{t \sqrt{({}^2k\, g)} + {}^2e\} + \&c. \\ y_2 = {}^1H_2 \cos.\{t \sqrt{({}^1k\, g)} + {}^1e\} + {}^2H_2 \cos.\{t \sqrt{({}^2k\, g)} + {}^2e\} + \&c. \\ \&c. = \&c. \end{array}\right\} \ldots (6)$$

where $^1H_1$, $^1H_2$, &c. must be deduced from β_1, β_2, &c. the original values of y_1, y_2, &c.

If the points have no initial velocities (i. e. when $t = 0$) we shall have $^1E = 0$, $^2E = 0$, &c.

We may have symmetrical oscillations in the following manner. If, of the quantities ^1E, ^2E, ^3E, &c. all vanish except one, for instance ^nE; we have

$$\left.\begin{array}{l} y_1 + {}^1\lambda\, y_2 + {}^1\lambda'\, y_3 + \&c. = 0 \\ y_1 + {}^2\lambda\, y_2 + {}^2\lambda'\, y_3 + \&c. = 0 \\ y_1 + {}^3\lambda\, y_2 + {}^3\lambda'\, y_3 + \&c. = 0 \\ - - - - - - - - - - \\ y_1 + {}^n\lambda\, y_2 + {}^n\lambda'\, y_3 + \&c. = {}^nE \cos. t \sqrt{({}^nk\, g)} \end{array}\right\} \ldots . (7)$$

omitting ^nE.

From the n — 1 of these equations, it appears that y_2, y_3, &c. are in a given ratio to y_1; and hence

$$y_1 + {}^n\lambda\, y_2 + {}^n\lambda'\, y_3 + \&c.$$

is a given multiple of y_1 and $= m\, y_1$ suppose. Hence, we have

$$m\, y_1 = {}^nE \cos. \surd\, ({}^nk\, g);$$

or, omitting the index n, which is now unnecessary,

$$m\, y_1 = E \cos. t \surd\, (k\, g).$$

Also if $y_2 = e_2\, y_1$,

$$m\, y_2 = E\, e_2 \cos. t \surd\, (k\, g)$$

and similarly for y_3 &c.

Hence, it appears that in this case the oscillations are *symmetrical.* All the points come into the vertical line at the same time, and move similarly, and contemporaneously on the two sides of it. The relation among the original ordinates β_1, β_2, β_3, &c. which must subsist in order that the oscillations may be of this kind, is given by the n — 1 equations (7),

$$\begin{aligned} \beta_1 + {}^1\lambda\, \beta_2 + {}^1\lambda'\, \beta_3 + \&c. &= 0 \\ \beta_1 + {}^2\lambda\, \beta_2 + {}^2\lambda'\, \beta_3 + \&c. &= 0 \\ \beta_1 + {}^3\lambda\, \beta_2 + {}^3\lambda'\, \beta_3 + \&c. &= 0 \\ \&c. &= \&c. \end{aligned}$$

These give the proportion of $\beta_1\, \beta_2$, &c; the arbitrary constant ${}^nE$, in the remaining equation, gives the *actual quantity* of the original displacement.

Also, we may take any one of the quantities ${}^1E$, ${}^2E$, ${}^3E$, &c. for that which does not vanish; and hence obtain, in a different way, such a system of n — 1 equations as has just been described. Hence, there are n different relations among $\beta_1\, \beta_2$, &c. or n different modes of arrangement, in which the points may be placed, so as to oscillate symmetrically.

(We might here also find these positions, which give symmetrical oscillations, by requiring the force in each of the ordinates $P_1\, M_1$, $P_2\, M_2$ to be as the distance; in which case the points P_1, P_2, &c. would all come to the vertical at the same time.

If the quantities $\surd\, {}^1k$, $\surd\, {}^2k$ have one common measure, there will be a time after which the pendulum will come into its original position. And it will describe similar successive cycles of vibrations. If these quantities be not commensurable, no portion of its motion will be similar to any preceding portion.)

The time of oscillation in each of these arrangements is easily known; the equation

$$m\, y_1 = {}^nE \cos. t \surd\, ({}^nk\, g)$$

shows that an oscillation employs a time

$$t = \frac{\pi}{\sqrt{({}^{n}k\, g)}}.$$

And hence, if all the roots ${}^{1}k$, ${}^{2}k$, ${}^{3}k$, &c. be different, the time is different for each different arrangement.

If the initial arrangement of the points be different from all those thus obtained, the oscillations of the pendulum may always be considered as compounded of n symmetrical oscillations. That is, if an imaginary pendulum oscillate symmetrically about the vertical line in a time

$$\frac{\pi}{\sqrt{({}^{1}k\, g)}};$$

and a second imaginary pendulum oscillate about the place of the first, considered as a fixed line, in the time

$$\frac{\pi}{\sqrt{({}^{2}k\, g)}};$$

and a third about the second, in the same manner, in the time

$$\frac{\pi}{\sqrt{({}^{3}k\, g)}};$$

and so on; the n^{th} pendulum may always be made to coincide perpetually with the real pendulum, by properly adjusting the amplitudes of the imaginary oscillations. This appears by considering the equations (6), viz.

$$y_1 = {}^{1}H_1 \cos. t \sqrt{({}^{1}k\, g)} + {}^{2}H_1 \cos. t \sqrt{({}^{2}k\, g)} + \&c.$$
$$\&c. = \&c.$$

This principle of the *coexistence of vibrations* is applicable in all cases where the vibrations are indefinitely small. In all such cases each set of symmetrical vibrations takes place, and affects the system as if that were the only motion which it experienced.

A familiar instance of this principle is seen in the manner in which the circular vibrations, produced by dropping stones into still water, spread from their respective centers, and cross without disfiguring each other.

If the oscillations be not all made in one vertical plane, we may take a horizontal ordinate z perpendicular to y. The oscillations in the direction of y will be the same as before, and there will be similar results obtained with respect to the oscillations in the direction of z.

We have supposed that the motion in the direction of x, the vertical axis, may be neglected, which is true when the oscillations are very small.

410. Ex. Let there be three bodies all equal (each $= \mu$), and also their distances a_1, a_2, a_3 all equal (each $= a$).

Here

$$p = 3\mu,\ p_2 = 2\mu,\ p_3 = \mu$$

and equations (3) become

$$a k = 5 - 2\lambda$$
$$a k \lambda = -2 + 3\lambda - \lambda'$$
$$a k \lambda' = -\lambda + \lambda'.$$

Eliminating k, we have

$$5\lambda - 2\lambda^2 = -2 + 3\lambda - \lambda',$$
$$5\lambda' - 2\lambda\lambda' = -\lambda + \lambda',$$

or

$$\lambda' = 2\lambda^2 - 2\lambda - 2,$$
$$4\lambda' - 2\lambda\lambda' = -\lambda$$
$$\therefore \lambda' = \frac{\lambda}{2\lambda - 4}$$
$$\therefore (2\lambda^2 - 2\lambda - 2)(2\lambda - 4) = \lambda$$

or

$$\lambda^3 - 3\lambda^2 + \frac{3}{4}\lambda + 2 = 0,$$

which may be solved by Trigonometrical Tables. We shall find three values of λ.

Hence, we have a value of λ' corresponding to each value of λ; and then by equations (7)

$$\left.\begin{array}{l}\beta_1 + {}^1\lambda\,\beta_2 + {}^1\lambda'\,\beta_3 = 0\\ \beta \ + {}^2\lambda\,\beta_2 + {}^2\lambda'\,\beta_3 = 0\end{array}\right\} \quad \dots\dots \quad (7)$$

whence we find β_2, β_3 in terms of β_1.

We shall thus find

$$\beta_2 = 2.295\,\beta_1$$

or

$$\beta_2 = 1.348\,\beta_1$$

or

$$\beta_2 = -.643\,\beta_1$$

according as we take the different values of λ.

And the times of oscillation in each case will be found by taking the value of

$$a k = 5 - 2\lambda;$$

that value of λ being taken which is not used in equation (7'). For the time of oscillation will be given by making

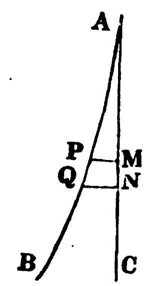
$$t\sqrt{(k g)} = \pi.$$

If the values of β_1, β_2, β_3 have not this initial relation, the oscillations

will be compounded in a manner similar to that described in the example for two bodies only.

411. *A flexible chain, of uniform thickness, hangs from a fixed point: to find its initial form, that its small oscillations may be symmetrical.*

Let A M, the vertical abscissa $= x$; M P the horizontal ordinate $= y$; A P $= s$, and the whole length A C $= a$;



$$\therefore \text{A P} = a - s.$$

And as before, the tension at P, when the oscillations are small, will be the weight of P C, and may be represented by $a - s$. This tension will act in the direction of a tangent at P, and hence the part of it in the direction P M will be

$$\text{tension} \times \frac{dy}{ds}$$

or

$$(a - s)\frac{dy}{ds}.$$

Now, if we take any portion P Q $= h$, we shall find the horizontal force at Q in the same manner. For the point Q, supposing ds constant

$$\frac{dy}{ds} \text{ becomes } \frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{d s_3} - \frac{h^2}{1.2}\,\&\text{c.}$$

(see 32).

Also, the tension will be $a - s + h$. Hence the horizontal force in the direction N Q, is

$$(a - s + h)\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \&\text{c.}\right)$$

Subtracting from this the force in P M, we have the force on P Q horizontally.

$$= (a - s)\left(\frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds^3}\cdot\frac{h^2}{1.2} + \&\text{c.}\right)$$
$$- h\left(\frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \&\text{c.}\right)$$

and the mass of P Q being represented by h, the accelerating force $\left(= \frac{\text{pressure} \times g}{\text{mass}}\right)$ is found. But since the different points of P Q move with different velocities, this expression is only applicable when h is indefinitely small. Hence, supposing Q to approach to and coincide with P, we have, when h vanishes

$$\text{accelerating force on P} = (a - s)\frac{d^2y}{ds^2} - \frac{dy}{ds}.$$

But since the oscillations are indefinitely small, x coincides with s and we have

$$\text{accelerating force on P} = (a-x)\frac{d^2 y}{d x^2} - \frac{d y}{d x}.$$

Now, in order that the oscillations may be symmetrical, this force must be in the direction P M, and proportional to P M, in which case all the points of A C, will come to the vertical A B at once. Hence, we must have

$$(a-x)\frac{d^2 y}{d x^2} - \frac{d y}{d x} = -k d y \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

k being some constant quantity to be determined.

This equation cannot be integrated in finite terms. To obtain a series let

$$y = A + B.(a-x) + C(a-x)^2 + \&c.$$

$$\therefore \frac{d y}{d x} = -B - 2C(a-x) - 3D(a-x)^2$$

$$\therefore \frac{d^2 y}{d x^2} = 1.2.C + 2.3D(a-x) + \&c.$$

Hence

$$0 = (a-x)\frac{d^2 y}{d x^2} - \frac{d y}{d x} + k y$$

gives

$$\begin{aligned} 0 = {} & 1.2.C(a-x) + 2.3D(a-x)^2 + \&c. \\ & + B + 2C(a-x) + 3D(a-x)^2 + \&c. \\ & + kA + kB(a-x) + kC(a-x)^2 + \&c. \end{aligned}$$

Equating coefficients; we have

$$\begin{aligned} B &= -kA, \\ 2^2 C &= -kB \\ 3^2 D &= -kC \\ \&c. &= \&c. \\ \therefore B &= kA \\ C &= \frac{k^2 A}{2^2} \\ D &= \frac{k^3 A}{2^2.3^2} \\ \&c. &= \&c. \end{aligned}$$

and

$$v = A\left\{1 - k(a-x) + \frac{k^2}{2^2}(a-x)^2 - \frac{k^3}{2^2.3^2}.(a-x) + \&c.\right\} \; . \; . \; (2)$$

Here

A is B C, the value of y when $x = a$. When $x = 0$, $y = 0$;

$$\therefore 1 - k\,a + \frac{k^2 a^2}{2^2} - \frac{k^3 a^3}{2^2 . 3^2} + \&c. = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From this equation (k) may be found. The equation has an infinite number of dimensions, and hence k will have an infinite number of values, which we may call

$${}^1k,\ {}^2k, \ldots {}^nk \ldots 1,$$

and these give an infinite number of initial forms, for which the chain may perform symmetrical oscillations.

The time of oscillation for each of these forms will be found thus. At the distance y, the force is k g y: hence by what has preceded, the time to the vertical is

$$\frac{\pi}{2\sqrt{(k\,g)}}$$

and the time of oscillation is

$$\frac{\pi}{\sqrt{(k\,g)}}.$$

(The greatest value of k a is about 1.44 (Euler Com. Acad. Petrop. tom. viii. p. 43). And the time of oscillation for this value is the same as that of a simple pendulum, whose length is $\frac{2}{3}$ a nearly.)

The points where the curve cuts the axis will be found by putting $y = 0$. Hence taking the value ${}^nk$ of k, we have

$$0 = 1 - {}^nk\,(a - x) + \frac{{}^nk^2 (a - x)^2}{2^2} + \frac{{}^nk^3 (a - x)^2}{2\ . 3^2} + \&c.$$

which will manifestly be verified, if

$${}^nk\,(a - x) = {}^1k\,a$$

or

$${}^nk\,(a - x) = {}^2k\,a$$

or

$${}^nk\,(a - x) = {}^3k\,a$$

$$\&c. = \&c.$$

because ${}^1k\,a$, ${}^2k\,a$, &c. are roots of equation (3).

That is if

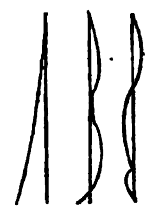
$$x = a\left(1 - \frac{{}^1k}{{}^nk}\right) \text{ or } = a\left(1 - \frac{{}^2k}{{}^nk}\right) \text{ or } = \&c.$$

Suppose ${}^1k$, ${}^2k$, ${}^3k$, &c. to be the roots in the order of their magnitude ${}^1k$ being the least.

Then if for ${}^nk$, we take ${}^1k$, all these values of x will be negative, and the curve will never cut the vertical axis below A.

If for ^{n}k, we take ^{1}k, all the values of x will be negative except the first; therefore, the curve will cut A B in one point. If we take ^{3}k, all the values will be negative except the two first, and the curve cuts A B in two points; and so on.

Hence, the forms for which the oscillations will be symmetrical, are of the kind thus represented.



And there are an infinite number of them, each cutting the axis in a different number of points.

If we represent equation (2) in this manner

$$y = A\,\varphi\,(k, x)$$

it is evident that

$$y = {}^1A\,\varphi\,({}^1k, x)$$
$$y = {}^2A\,\varphi\,({}^2k, x)$$
$$\&c. = \&c.$$

will each satisfy equation (1). Hence as before, if we put

$$y = {}^1A\,\varphi\,({}^1k, x) + {}^2A\,\varphi\,({}^2k, x) + \&c.$$

and if ^{1}A, ^{2}A, &c. can be so assumed that this shall represent a given initial form of the chain, its oscillations shall be compounded of as many coexisting symmetrical ones, as there are terms ^{1}A, ^{2}A, &c.

We shall now terminate this long digression upon constrained motion. The reader who wishes for more complete information may consult Whewell's Dynamics, one of the most useful and elegant treatises ever written, the various speculations of Euler in the work above quoted, or rather the comprehensive methods of Lagrange in his Mecanique Analytique.

We now proceed to simplify the text of this Xth Section.

412. PROP. L. First, S R Q is formed by an entire revolution of the generating circle or wheel, whose diameter is O R, upon the globe S O Q.

413. Secondly, by taking

$$CA : CO :: CO : CR$$

we have

$$CA : CO :: CA - CO : CO - CR$$
$$:: AO \quad : OR$$

and therefore if C S be joined and produced to meet the exterior globe in D, we have also

$$AD : SO (:: CA : CO) :: AO : OR.$$

But

$$SO = \text{the semi-circumference of the wheel } OR = \frac{\pi . OR}{2}.$$

$$\therefore A D = \frac{\pi . A O}{2} = \tfrac{1}{2} \text{ the circumference of the wheel whose diameter is}$$

A O. That is S is the vertex of the Hypocycloid A S, and A S is perpendicular in S to C S. But O S is also perpendicular to C S. Therefore A S touches O S in S, &c.

414. *The similar figures* A S, S R.]

By 39 it readily appears that Hypocycloids are similar when

$$R : r :: R' : r'$$

R and r being the radii of the globe and wheel; that is when

$$C A : A O :: C O : O R$$

or when

$$C A : C O :: C O : C R$$

$$\therefore A S, S R \text{ are similar}$$

415. V B, V W *are equal to* O A, O R.]

If B be not in the circumference A D let C V meet it in B'. Then V P being a tangent at P, and since the element of the curve A P is the same as would be generated by the revolution of B' P around B' as a center, and ∴ B' P is perpendicular both to the curve and its tangent P V, therefore P B, P B' and ∴ B, B' coincide. That is

$$V B = O A.$$

Also if the wheel O R describes O V whilst A O describes A B, the angular velocity B P in each must be the same, although at first, viz. at O and A, they are at right angles to each other. Hence when they shall have arrived at V and B their distances from C B must be complements of each other. But

$$\angle T V W = B V P = \frac{\pi}{2} - P B V$$

∴ T V is a chord in the wheel O R, and

$$\therefore V W = O R.$$

See also the Jesuits' note.

OTHERWISE.

416. Construct the curve S P, to which the radius of curvature to every point of S R Q is a tangent; or which is the same, find S A the Locus of the Centers of Curvature to S R Q.

Hence is suggested the following generalization of the Problem, viz.

417. *To make a body oscillate in any given curve.*

Let S R Q (Newton's fig.) the given curve be symmetrical on both sides

of R. Then if x, y be the rectangular coordinates referred to the vertex R, and α, β those of the centers of curvature (P) we have

$$r^2 = PT^2 = (y-\beta)^2 + (x-\alpha)^2.$$

Hence, the contact being of the second order (74)

$$x - \alpha + (y-\beta)\frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$1 + \frac{dy^2}{dx^2} + (y-\beta)\frac{d^2y}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two equations by means of that of the given curve, will give us β in terms of α, or the equation to the Locus of the centers of curvature.

Let S A be the Locus corresponding to S R, and A Q the other half. Then suspending a body from A attached to a string whose length is R A, when this string shall be stretched into any position A P T, it is evident that P being the point where the string quits the locus is a tangent to it, and that T is a point in S R Q.

Ex. 1. *Let* S R Q *be the common parabola.*

Here

$$y^2 = 2ax$$

$$\therefore \frac{dy}{dx} = \frac{a}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2}\cdot\frac{dy}{dx} = -\frac{a^2}{y^3} = -\frac{a}{2xy}$$

$\therefore$ substituting we get

$$x - \alpha + (y-\beta)\cdot\frac{a}{y} = 0$$

and

$$1 + \frac{a}{2x} - (y-\beta)\cdot\frac{a}{2xy} = 0$$

$$\therefore x - \alpha + \frac{a}{y}\left(1 + \frac{a}{2x}\right)\cdot\frac{2xy}{a} = 0 = 3x - \alpha + a$$

or

$$\left.\begin{array}{l}\alpha = 3x + a \\ \therefore \beta = -\dfrac{2xy}{a}\end{array}\right\}$$

and

But

$$y^2 = 2ax$$

$$\therefore \beta^2 = \frac{4x^2 \cdot y^2}{a^2} = \frac{8x^3}{a}$$

$$= \frac{8}{a} \times \frac{(\alpha - a)^3}{27} = \frac{8}{27 a} \cdot (\alpha - a)^3 \quad . \quad . \quad . \quad (3)$$

Now when $\beta = 0$, $\alpha = a$; which shows that A R the length of the string must equal a. Also making A the origin of abscissas, that is, augmenting α by a, we have

$$\beta^2 = \frac{8}{27 a} \times \alpha'^3$$

the equation to the semicubical parabola A S, A Q, which may be traced by the ordinary rules (35, &c.); and thereby the body be made to oscillate in the common parabola S Q R.

Ex. 2. *Let* S R Q *be an ellipse.*

Then, referring to its center, instead of the vertex,

$$y^2 = \frac{b^2}{a^2} \cdot (a^2 - x^2)$$

or

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

$$\therefore a^2 y \frac{d y}{d x} + b^2 x = 0$$

and

$$a^2 y \frac{d^2 y}{d x} + a^2 \frac{d y^2}{d x^2} + b^2 = 0.$$

These give

$$\frac{d y}{d x} = - \frac{b^2 x}{a^2 y}$$

and

$$\frac{d^2 y}{d x^2} = - \frac{b^4}{a^2 y^3}.$$

Hence

$$\alpha = \frac{(a^2 - b^2) x^3}{a^4}$$

and

$$\beta = - \frac{(a^2 - b^2) y^3}{b^4}.$$

Hence substituting the values of y and x in

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

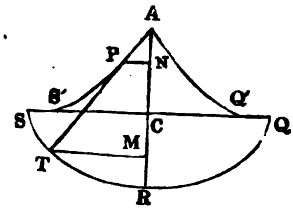
we get

$$\left(\frac{\beta b}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{\alpha a}{a^2 - b^2}\right)^{\frac{2}{3}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

the equation to the Locus of the centers of curvature.

In the annexed figure let

$$S\,C = b,\ C\,R = a$$
$$C\,M = x,\ T\,M = y.$$



Then

$$P\,N = \beta,\ C\,N = \alpha.$$

And to construct A S' by points, first put

$$\beta = 0$$

whence by equation (a)

$$\alpha = \pm \frac{a^2 - b^2}{a}$$

the value of A C. Let

$$\alpha = 0$$

then

$$\beta = \pm \frac{a^2 - b^2}{b}$$

the value of S' C or C Q'.

Hence to make a body oscillate in the semi-ellipse S R Q we must take a pendulum of the length A R, (part = A P S' flexible, and part = S S' rigid; because S S' is horizontal, and no string however stretched can be horizontal—see Whewell's Mechanics,) and suspend it at A. Then A P being in contact with the Locus A S', P T will also touch A S in P, &c. &c.

Ex. 3. *Let* S R Q *be the common cycloid;*

The equation to the cycloid is

$$\frac{d\,y}{d\,x} = \sqrt{\frac{2\,r - y}{y}} = \sqrt{\left(\frac{2\,r}{y} - 1\right)}$$

$$\therefore \frac{d^2 y}{d\,x^2} = -\frac{r}{y^2}$$

whence it is found that

$$\left.\begin{array}{l}\alpha = x + 2\sqrt{(2\,r\,y - y^2)} \\ \beta = -y\end{array}\right\}.$$

Hence

$$\frac{d\,\alpha}{d\,x} = \frac{2\,r - y}{y}$$

and

$$\frac{d\,\beta}{d\,x} = -\frac{d\,y}{d\,x} = -\sqrt{\frac{2\,r - y}{y}}$$

$$\therefore \frac{d\,\beta}{d\,\alpha} = -\sqrt{\frac{y}{2\,r - y}} = -\sqrt{\frac{-\beta}{2\,r + \beta}}$$

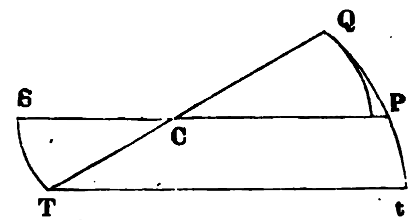
which is also the equation of a cycloid, of which the generating circle is

precisely the same as the former, the only difference consisting in a change of sign of the ordinate, and of the origin of the abscissæ.

The rest of this section is rendered sufficiently intelligible by the Notes of P. P. Le Seur and Jacquier; and by the ample supplementary matter we have inserted.

SECTION XI.

417. Prop. LVII. Two bodies attracting one another, describe round each other and round the center of gravity similar figures.



Since the mutual actions will not affect the center of gravity, the bodies will always lie in a straight line passing through C, and their distances from C will always be in the same proportion.

$$\therefore \text{SC} : \text{TC} :: \text{PC} : \text{QC}$$

and

$$\angle \text{SCT} = \text{QCP}.$$

$\therefore$ the figures described round each other are similar.

Also if T t be taken = S P, the figure which P seems to describe round S will be t Q, and

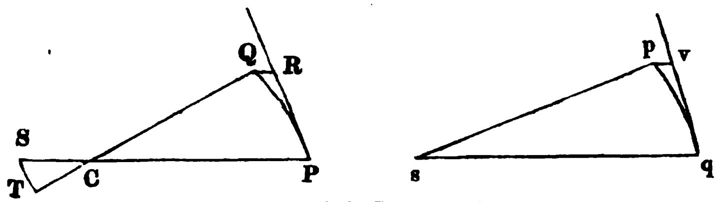
$$\begin{aligned} \text{Tt} : \text{TQ} &:: \text{SP} : \text{TQ} \\ &:: \text{CP} : \text{CQ} \end{aligned}$$

and

$$\angle \text{tTQ} = \text{PCQ}.$$

$\therefore$ the figures t Q, P Q, are similar; and the figure which S seems to describe round P is similar, and equal to the figure which P seems to describe round S.

418. Prop. LVIII. If S remained at rest, a figure might be described by P round S, similar and equal to the figures which P and S seem to describe round each other, and by an equal force.



Curves are supposed similar and Q R, q r indefinitely small. Let P and p be projected in directions P R, p r (making equal angles C P R, s p r) with such velocities that

$$\frac{V}{v} = \frac{\sqrt{S}}{\sqrt{S+P}} = \frac{\sqrt{CP}}{\sqrt{sp}} = \frac{\sqrt{PQ}}{\sqrt{pq}}.$$

Then $\left(\text{since } dt = \frac{ds}{v}\right)$

$$\frac{T}{t} = \frac{PQ}{pq} \cdot \frac{\sqrt{pq}}{\sqrt{PQ}} = \frac{\sqrt{PQ}}{\sqrt{pq}} = \frac{\sqrt{QR}}{\sqrt{qr}}.$$

But in the beginning of the motion $f = \frac{S}{\frac{gt^2}{2}}$

$$\therefore \frac{F}{f} = \frac{QR}{qr} \cdot \frac{qr}{QR} = \frac{1}{1}.$$

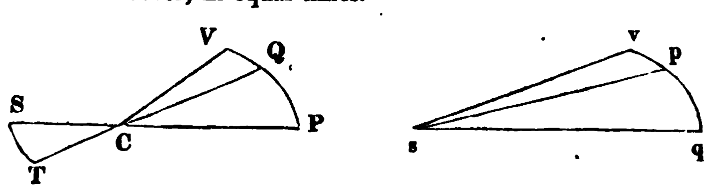
The same thing takes place if the center of gravity and the whole system move uniformly forward in a straight line in fixed space.

419. COR. 1. If $F \propto D$, the bodies will describe round the common center of gravity, and round each other, concentric ellipses, for such would be described by P round S at rest with the same force.

Conversely, if the figures be ellipses concentric, $F \propto D$.

420. COR. 2. If $F \propto \frac{1}{D^2}$ the figures will be conic sections, the foci in the centers of force, and the converse.

421. COR. 3. Equal areas are described round the center of gravity, and round each other, in equal times.



422. COR. 3. Otherwise. Since the curves are similar, the areas, bounded by similar parts of the curves, are similar or proportional.

$\therefore spq : CPQ :: sp^2 : CP^2 :: (S+P)^2 : s^2$ in a given ratio;

and T. through s p q : T. through C P Q : : $\sqrt{\overline{S+P}}$: $\sqrt{S}$, in a given ratio
and $\therefore$: : T. through spv : T. through CPV
$\therefore$ T. through C P Q : T. through C P V : : T. through s p q : T. through s p v
: : s p q : s p v (by Sect. II.)
: : C P Q : C P V

$\therefore$ the areas described round C are proportional to the times, and the areas described round each other in the same times, which are similar to the areas round C, are also proportional to the times.

423. Prop. LIX. The period in the figure described in last Prop. : the period round C : : $\sqrt{S+P}$: $\sqrt{S}$; for the times through similar arcs p q, P Q, are in that proportion.

424. Prop. LX. The major axis of an ellipse which P seems to describe round S in motion $\left(\text{Force} \propto \frac{1}{D^2}\right)$: major axis of an ellipse which would be described by P *in the same time* round S at rest : : $\overline{S+P}$: first of two mean proportionals between $\overline{S+P}$ and S.

Let A = major axis of an ellipse described (or seemed to be described) round S in motion, and which is similar and equal to the ellipse described in Prop. LVIII.

Let x = major axis of an ellipse which would be described round S at rest in the same time.

$$\therefore \frac{\text{period in ellipse round S in motion}}{\text{period in same ellipse round S at rest}} = \frac{\sqrt{S}}{\sqrt{S+P}} \text{(Prop. LIX)}$$

and by Sect. III,

$$\frac{\text{period in ellipse round S at rest}}{\text{period in required ellipse round S at rest}} = \frac{A^{\frac{3}{2}}}{x^{\frac{3}{2}}}$$

$$\therefore \frac{\text{period in ellipse round S in motion}}{\text{period in required ellipse round S at rest}} = \frac{A^{\frac{3}{2}}\sqrt{S}}{x^{\frac{3}{2}}\sqrt{S+P}}$$

but these periods are to be equal,

$$\therefore A^3 S = x^3 . \overline{S+P}$$

$\therefore$ A : x : : $\sqrt[3]{S+P}$: $\sqrt[3]{S}$: : S + P : first of two mean proportionals

(for if a, a r, a r^2, a r^3, be proportionals, $\sqrt[3]{a}$: $\sqrt[3]{a r^3}$: : a : a r.)

425. At what mean distance from the earth would the moon revolve round the earth at rest, in the same time as she now revolves round the earth in motion? This is easily resolved.

426. Prop. LXI. The bodies will move as if acted upon by bodies at the center of gravity with the same force, and the law of force with re-

spect to the distances from the center of gravity will be the same as with respect to the distances from each other.

For the force is always in the line of the center of gravity, and $\therefore$ the bodies will be acted upon as if it came from the center of gravity.

And the distance from the center of gravity is in a given ratio to the distance from each other, $\therefore$ the forces which are the same functions of these distances will be proportional.

427. PROP. LXII. *Problem of two bodies with no initial Velocities.*

$F \propto \frac{1}{D^2}$. Two bodies are let fall towards each other. Determine the motions.

The center of gravity will remain at rest, and the bodies will move as if acted on by bodies placed at the center of gravity, (and exerting the same force at any given distance that the real bodies exert),

$\therefore$ the motions may be determined by the 7th Sect.

428. PROP. LXIII. *Problem of two bodies with given initial Velocities.*

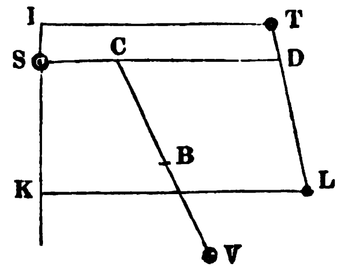
$F \propto \frac{1}{D^2}$. Two bodies are projected in given directions, with given velocities. Determine the motions.

The motion of the center of gravity is known from the velocities and directions of projection. Subtract the velocity of the center of gravity from each of the given velocities, and the remainders will be the velocities with which the bodies will move in respect of each other, and of the center of gravity, as if the center of gravity were at rest. Hence since they are acted upon as if by bodies at the center of gravity, (whose magnitudes are determined by the equality of the forces), the motions may be determined by Prop. XVII, Sect. III, (velocities being supposed to be acquired down the finite distance), if the directions of projection do not tend to the center, or by Prop. XXXVII, Sect. VII, if they tend to or directly from the center. Thus the motions of the bodies with respect to the center of gravity will be determined, and these motions compounded with the uniform motion of the center of gravity will determine the motions of the bodies in absolute space.

429. PROP. LXIV. $F \propto D$, determine the motions of any number of bodies attracting each other.

T and L will describe concentric ellipses round D.

Now add a third body S.



Attraction of S on T may be represented by the distance T S, and on L by L S, (attraction at distance being 1) resolve T S, L S, into T D, D S; L D, D S, whereof the parts T D, L D, being in given ratios to the whole, T L, L T, will only increase the forces with which L and T act on each other, and the bodies L and T will continue to describe ellipses (as far as respects these new forces) but with accelerated velocities, (for in similar parts of similar figures $V^2 \propto F.R$ Prop. IV. Cor. 1 and 8.) The remaining forces D S, and D S, being equal and parallel, will not alter the relative motions of the bodies L and T, ∴ they will continue to describe ellipses round D, which will move towards the line I K, but will be impeded in its approach by making the bodies S and D (D being T + L) describe concentric ellipses round the center of gravity C, being projected with proper velocities, in opposite and parallel directions. Now add a fourth body V, and all the previous motions will continue the same, only accelerated, and C and V will describe ellipses round B, being projected with proper velocities.

And so on, for any number of bodies.

Also the periods in all the ellipses will be the same, for the accelerating force on $T = L.TL + S.TD = (T+L).TD + S.TD = (T+L+S).TD$, i. e. when a third body S is added, T is acted on as if by the sum of the three bodies at the distance T D, and the accelerating force on D towards $C = S.SD = S.CS + S.DC = (T+L).DC + S.DC = (T+L+S).DC$.

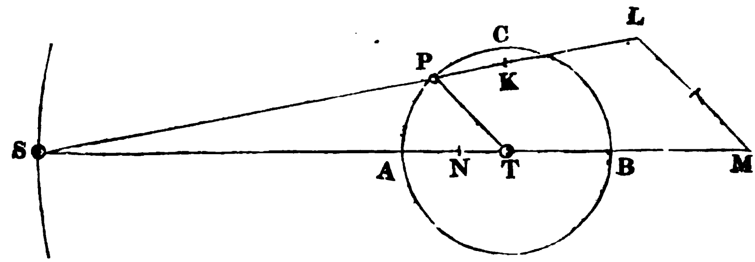
∴ accelerating force on T towards D : do. on D towards C : : T D : D C

∴ the absolute accelerating forces on T and D are equal, or T and D move as if they revolved round a common center, the absolute force the same, and varying as the distance from the center, i. e. they describe ellipses, in the same periods.

Similarly when a fourth body V is added, T, L, D, S, C, and V, move as if the four bodies were placed at D, C, B, i. e. as if the absolute forces were the same, and with forces proportional to their respective distances from the centers of gravity, and ∴ in equal periods.

And so on, for any number of bodies.

430. PROP. LXVI. S and P revolve round T, S in the exterior orbit, P in the interior,



$F \propto \frac{1}{D^2}$, find when P will describe round T an orbit nearest to the ellipse, and areas most nearly proportional to the times.

(1st.) Let S, P, revolve round the greatest body T *in the same plane.*

Take K S for the force of S on P at the mean distance S K,

and

$$L S = S K \cdot \frac{S K^2}{S P^2} = \text{force at P},$$

resolve L S into L M, M S,

L M is parallel to P T, and $\therefore$ tends to the center T, $\therefore$ P will continue to describe areas round T proportional to the times, as when acted on only by P T, but since L M does not $\propto \frac{1}{P T^2}$, the sum of L M and P T will not $\propto \frac{1}{P T^2}$, $\therefore$ the form of the elliptic orbit P A B will be disturbed by this force, L M, M S neither tends from P to the center T, nor $\propto \frac{1}{P T^2}$, $\therefore$ from the force M S both the proportionality of areas to times, and the elliptic form of the orbit, will be disturbed, and the elliptic form on two accounts, because M S does not tend to C, and because it does not $\propto \frac{1}{P T^2}$.

$\therefore$ the areas will be most proportional to the times, when the force M S is least, and the elliptic form will be most complete, when the forces M S, L M, but particularly L M, are least.

Now let the force of S on T = N S, then this first part of the force M S being common to P and T will not affect their mutual motions, $\therefore$ the

disturbing forces will be least when L M, M N, are least, or L M remaining, when M N is least, i. e. when the forces of S on P and T are nearly equal, or S N nearly = S K.

(2dly) Let S and P revolve round T *in different planes.*

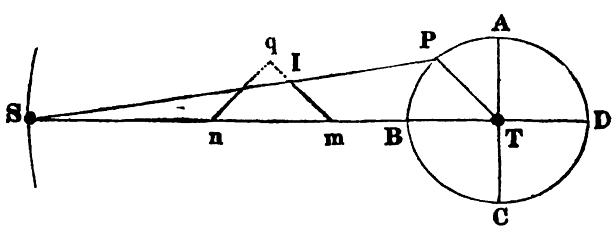
Then L M will act as before.

But M N acting parallel to T S, when S is not in the line of the Nodes, (and M N does not pass through T), will cause a disturbance not only in the longitude as before, but also in the latitude, by deflecting P from the plane of its orbit. And this disturbance will be least, when M N is least, or S N nearly = S K.

431. Cor. 1. If more bodies revolve round the greatest body T, the motion of the inmost body P will be least disturbed when T is attracted by the others equally, according to the distances, as they are attracted by each other.

432. Cor. 2. In the system of T, if the attractions of any two on the third be as $\frac{1}{D^2}$, P will describe areas round T with greater velocity near conjunction and opposition, than near the quadratures.

433. To prove this, the following investigation is necessary.



Take l S to represent the attraction of S on P,

n S ———————————— T,

Then the disturbing forces are l m (parallel to P T) and m n.

Now

$$S\,l = \frac{S}{S\,P^2}\ \left(\text{force} \propto \frac{1}{D^2}\right),$$

$$= \frac{S}{R^2 - 2\,R\,r\cos.A + r^2},\ (R = S\,T, r = P\,T)\ A = \angle S\,T\,P$$

$$\therefore S\,m = S\,l.\frac{S\,T}{S\,P} = \frac{S.R}{(R^2 - 2\,R\,r\cos.A + r^2)\sqrt{R^2 - 2\,R\,r\cos.A + r^2}}$$

$$= \frac{S.R}{R^3\left(1 - \frac{2\,r\cos.A}{R} + \frac{r^2}{R^2}\right)\sqrt{1 - \frac{2\,r\cos.A}{R} + \frac{r^2}{R^2}}}$$

$$= \frac{S}{R^2}\left(1 - \frac{2r}{R}\cos. A + \frac{r^2}{R^2}\right)^{-\frac{3}{2}}$$

$$= \frac{S}{R^2}\left(1 + \frac{3}{2}\left(\frac{2r}{R}\cos. A - \frac{r^2}{R^2}\right) + \frac{3.5}{2.4}.\left(\frac{2r\cos. A}{R} - \frac{r^2}{R^2}\right)^2 \&c.\right)$$

$$= \frac{S}{R^2}\left(1 + \frac{3r}{R}\cos. A - \left(\frac{3}{2} - \frac{3.5}{2.4}\cos.^2 A\right)\frac{r^2}{R^2}\&c.\right)$$

$$= \frac{S}{R^2}\left(1 + \frac{3r.\cos. A}{R}\right),$$

where R is indefinitely great with respect to r.

Also

$$m n = S m - S n = \frac{S}{R^2}\left(1 + \frac{3r\cos. A}{R}\right) - \frac{S}{R^2} = \frac{S.3r\cos.}{R^3},$$

ultimately

$$\text{and } l m = S l.\frac{r}{R} = \frac{S.r}{R}(R^2 - 2Rr\cos. A + r^2)$$

$$= \frac{S.r}{R}.(R^2 - 2Rr\cos. A + r^2)^{-1}$$

$$= \frac{S.r}{R^3} + \frac{S.2r^2}{R^4}, \&c.$$

$$= \frac{S.r}{R^3} \text{ ultimately.}$$

434. Call l m the addititious force
and m n the ablatitious force
and m n = l m 3 cos. A.
Resolve m n into m q, q n.

The part of the ablatitious force which acts in the direction m q = m n . cos. A

$$= \frac{3.S.r.\cos.^2 A}{R^3} = \text{central ablatitious force.}$$

$$\text{The tangential part} = m n.\sin. A = \frac{3Sr}{R^3}.\sin. A.\cos. A$$

$$= \frac{3}{2}.\frac{S.r}{R^3}.\sin. 2A = \text{tangential ablatitious force}$$

$$\therefore \text{the whole force in the direction } PT = l m - m q = \frac{S.r}{R^3} - \frac{3.S.r.\cos.^2 A}{R^3}$$

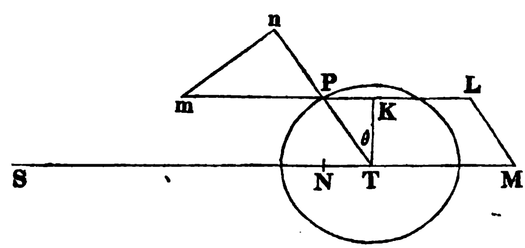
$$= \frac{S.r}{R^3}(1 - 3\cos.^2 A) \text{ and the}$$

$$\text{whole force in the direction of the Tangent} = q n = \frac{3}{2}.\frac{S.r}{R^3}.\sin. 2A.$$

435. Hence COR. 2. is manifest, for of the four forces acting on P, the

three first, namely, attraction of T, addititious force, and central ablatitious force, do not disturb the equable description of areas, but the fourth or tangential ablatitious force does, and this is + from A to B, — from B to C, + from C to D, — from D to A. ∴ the velocity is accelerated from A to B, and retarded from B to C, ∴ it is greatest at B. Similarly it is a maximum at D. And it is a minimum at A and C. This is Cor. 3.

436. *To otherwise calculate the central and tangential ablititious forces.*



On account of the great distance of S, S M, P L may be considered parallel, and

$$\therefore \text{P T} = \text{L M, and S P} = \text{S K} = \text{S T}.$$

∴ the ablatitious force $= 3\,\text{P T}.\sin.\theta = 3\,\text{P K}$.

Take P m = 3 P K, and resolve it into P n, n m.

$\text{P n} = \text{P m}.\sin.\theta = 3\,\text{P T}.\sin.^2\theta =$ central ablatitious force

$$= 3\,\text{P T}.\frac{1 - \cos.2\theta}{2}$$

$\text{n m} = \text{P m}.\cos.\theta = 3\,\text{P T}.\sin.\theta\cos.\theta = \frac{3}{2}.\text{P T}.\sin.2\theta =$ tangential ablatitious force.

The same conclusions may be got in terms of l m from the fig. in Art 433, which would be better.

437. *Find the disturbing force on* P *in the direction* P T.

This = (addititious + central ablatitious) force $= \text{l m} + 3\,\text{l m}.\sin.^2\theta$

$$= \text{l m} - 3\,\text{l m}\left(\frac{1 - \cos.2\theta}{2}\right)$$

$$= -\text{l m}\left(\frac{1 - 3\cos.2\theta}{2}\right).$$

438. *To find the mean disturbing force of* S *during a whole revolution in the direction* P T.

Let P T at the mean distance = m, then $-\text{l m}\left(\frac{1 - 3\cos.2\theta}{2}\right)$

$= -\frac{1\,m}{2} = -\frac{m}{2}$ since cos. $2\,\theta$ is destroyed during a whole revolution.

439. The disturbing forces on P are

(1) addititious $= \frac{S \cdot r}{R^3} = A$.

(2) ablatitious $= 3 \cdot A \cdot \sin.\ \theta$

which is (1) tangential ablatitious force $\frac{3 \cdot A}{2} \cdot \cos.\ 2\,\theta$

and (2) central ablatitious force $= 3\,A \cdot \frac{1 - \cos.\ 2\,\theta}{2}$

$\therefore$ whole disturbing force in the direction P T $= A - \frac{3\,A}{2} + \frac{3\,A}{2} \cdot \cos.\ 2\,\theta$

$$= -\frac{A}{2} + \frac{3\,A}{2} \cdot \cos.\ 2\,\theta.$$

But in a whole revolution cos. $2\,\theta$ will destroy itself, $\therefore$ the whole disturbing force in the direction P T in a complete revolution is ablatitious and $= \frac{1}{2}$ addititious force.

The whole force in the direction P T $= \frac{S \cdot r}{R^3}(1 - 3\sin.^2\theta)$ (Art. 433)

$$= \frac{S\,r}{R^3}\left(1 - \frac{3}{2}(1 - \cos.\ 2\,\theta)\right)$$

multiply this by d.θ, and the integral $= \frac{S \cdot r}{R^3}\left(\theta - \frac{3}{2}\theta + \frac{3}{4} \cdot \sin.\ 2\,\theta\right)$

$=$ sum of the disturbing forces; and this when $\theta = \pi$ becomes $-\frac{S \cdot r}{R^3} \cdot \frac{\pi}{2}$.

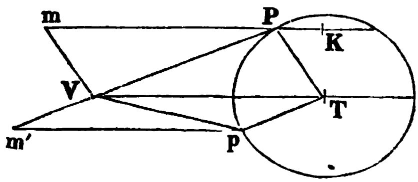
This must be divided by π, and it gives the mean disturbing force acting on P in the direction of radius vector $= -\frac{1}{2}\frac{S \cdot r}{R^3}$.

440. The 2d COR. will appear from Art. 433 and 434.

For the tangential ablatitious force $= \frac{3}{2} \cdot \sin\ 2\,\theta \cdot \times$ addititious force,

$\therefore$ this force will accelerate the description of the areas from the quadratures to the syzygies and retard it from the syzygies to the quadratures, since in the former case sin. $2\,\theta$ is $+$, and in the latter $-$.

441. COR. 3 is contained in COR. 2. (Hence the Variation in astronomy.)

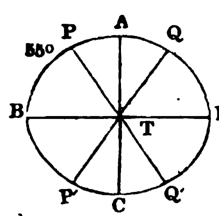


442. P V is equivalent to P T, T V, and accelerates the motion; p V is equivalent to p T, T V, and retards the motion.

443. Cor. 4. Cæt. par., the curve is of greater curvature in the quadratures than in the syzygies.

For since the velocity is greatest in the syzygies, (and the central ablatitious force being the greatest, the remaining force of P to T is the least) the body will be less deflected from a right line, and the orbit will be less curved. The contrary takes place in the quadratures.

444. The whole force from S in the direction $\text{P T} = \frac{\text{S} \cdot \text{r}}{\text{R}^3} (1 - 3 \sin.^2 \theta)$ (see 433) and the force from T in the direction $\text{P T} = \frac{\text{T}}{\text{r}^2}$.



$\therefore$ the whole force in the direction $\text{P T} = \frac{\text{T}}{\text{r}^2} + \frac{\text{S} \cdot \text{r}}{\text{R}^3} (1 - 3 \sin.^2 \theta)$

and at A this becomes $\frac{\text{T}}{\text{r}^2} + \frac{\text{S} \cdot \text{r}}{\text{R}^3}$

at B . . . . $\frac{\text{T}}{\text{r}^2} - \frac{2 \cdot \text{S} \cdot \text{r}}{\text{R}^3}$

at C . . . . $\frac{\text{T}}{\text{r}^2} + \frac{\text{S} \cdot \text{r}}{\text{R}^3}$

at D . . . . $\frac{\text{T}}{\text{r}^2} - \frac{2\,\text{S} \cdot \text{r}}{\text{R}^3}$

(for though sin. 270 is —, yet its syzygy is +).

Thus it appears that on two accounts the orbit is more curved in the quadratures than in the syzygies, and assumes the form of an ellipse at the major axis A C.

∴ the body is at a greater distance from the center in the quadratures than in the syzygies, which is Cor. 5.

445. COR. 5. Hence the body P, cæt. par., will recede farther from T in the quadratures than in the syzygies; for since the orbit is less curved in the syzygies than in the quadratures, it is evident that the body must be farther from the center in the quadratures than in the syzygies.

446. COR. 6. The addititious central force is greater than the ablatitious from Q′ to P, and from P′ to Q, but less from P to P′, and from Q to Q′, ∴ on the whole, the central attraction is diminished. But it may be said, that the areas are accelerated towards B and D, and ∴ the time through P P′ may not exceed the time through P′ Q, or the time through Q Q′ exceed that through Q′ P. But in all the corollories, since the errors are very small, when we are seeking the quantity of an error, and have ascertained it without taking into account some other error, there will be an error in our error, but this error in the error will be an error of the second order, and may ∴ be neglected.

The attraction of P to T being diminished in the course of a revolution, the absolute force towards T is diminished, (being diminished by the mean disturbing force $-\frac{1}{2}\frac{S r}{R^3}$, 439,) ∴ the period which $\propto \frac{r^{\frac{3}{2}}}{\sqrt{f}}$, is increased, supposing r constant.

But as T approaches S (which it will do from its higher apse to the lower) R is diminished, the disturbing force $\left(\text{which involves } \frac{r}{R}\right)$ will be increased, and the gravity of P to T still more diminished, and ∴ r will be increased; ∴ on both accounts (the diminution of f and increase of r) the period will be increased.

(Thus the period of the moon round the earth is shorter in summer than in winter. Hence the Annual equation in astronomy.)

When T recedes from S, R is increased, and the disturbing force diminished and r diminished. ∴ the period will be diminished (not in comparison with the period round T if there were no body S, but in comparison with what the period was before, from the actual disturbance.)

447. COR. 6. The whole force of P to T in the quadratures $=\frac{T}{r^2}+\frac{S \cdot r}{R^3}$

———————— the syzygies $=\frac{T}{r^2}-\frac{2 S r}{R^3}$

∴ on the whole the attraction of P to T is diminished in a revolution.

For the ablatitious force in the syzygies equals twice the addititious force in the quadratures.

At a certain point the ablatitious force = the addititious; when

$$1 = 3\sin.^2\theta$$

or

$$\sin.\theta = \frac{1}{\sqrt{3}}$$

and

$$A = 55^\circ, \&c.$$

(the whole force being then $= \frac{\Gamma}{r^2}$.)

Up to this point from the quadratures the addititious force is greater than the ablatitious force, and from this point to one equally distant from the syzygies on the other side, the ablatitious is greater than the addititious; $\therefore$ in a whole revolution P's gravity to T is diminished.

Again since T alternately approaches to and recedes from S, the radius P T is increased when T approaches S, and the period $\propto \frac{r^{\frac{3}{2}}}{\sqrt{\text{absolute force}}}$ and since f is diminished, and $\therefore$ r increased, $\therefore$ the periodic time is increased on both accounts, (for f is diminished by the increase of the disturbing forces which involve $\frac{r}{R}$.) If the distance of S be diminished, the absolute force of S on P will be increased. $\therefore$ the disturbing forces which $\propto \frac{1}{D^2}$ from S are increased, and P's gravity to T diminished, and $\therefore$ the periodic time is increased in a greater ratio than $r^{\frac{3}{2}}$ (because of the diminution of f in the expression $\frac{r^{\frac{3}{2}}}{\sqrt{f}}$) and when the distance of S is increased, the disturbing force will be diminished, (but still the attraction of P to T will be diminished by the disturbance of S) and r will be decreased, $\therefore$ the period will be diminished in a less ratio than $r^{\frac{3}{2}}$.

448. Cor. 7. To find the effect of the disturbing force on the motion of the apsides of P's orbit during one whole revolution.

Whole force in the direction $P T = \frac{T}{r^2} + \frac{S \cdot r}{R^3}(1 - 3\cos.^2 A)$

$= \frac{T}{r^2} + T \cdot c \cdot r$, (if $T \cdot c = \frac{S}{R^3}(1 - 3\cos.^2 A)$) $= \frac{T r + T \cdot c \cdot r^4}{r^3}$,

$\therefore$ the $\angle$ between the apsides $= 180 \quad \frac{1+c}{1+4c}$ by the IXth Sect. which is less than 180 when c is positive, i. e. from Q' to P and from P' to P,

(fig. (446,)) and greater than 180 when c is negative, i. e. from P to P′ and from Q to Q′,

$\therefore$ upon the whole the apsides are progressive, (regressive in the quadratures and progressive in the syzygies);

$$\text{force} = \frac{T}{r^2} - \frac{3\,S\,r}{R^3} = \text{force in conjunction}$$

$$\frac{T}{r'^2} - \frac{3 \cdot S\,r'}{R^3} = \text{force in opposition}$$

Now

$$\frac{R^3 T - 3\,S\,r^3}{r^2 R^3} \text{ and } \frac{R^3 T - 3\,S\,r'^3}{r'^2 R^3}$$

$$\text{differ most from } \frac{1}{r^2} \text{ and } \frac{1}{r'^2}$$

when r is least with respect to r′,

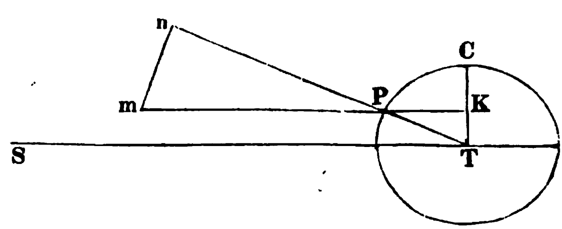
which is the case when the Apsides are in the syzygies.

But

$$\frac{R^3 T + S\,r^3}{r^2 R^3} \quad \frac{R^3 T + S\,r'^3}{r'^2 R^3}$$

differ least from $\frac{1}{r^2}$ and $\frac{1}{r'^2}$ when r is most nearly equal to r′,

449. COR. 7. Ex. Find the angle from the quadratures, when the apses are stationary.



Draw P m parallel to T S, and = 3 P K, m n perpendicular to T P, resolve P m into P n, n m, whereof n m neither increases nor diminishes the accelerating force of P to T, but P n lessens that force, $\therefore$ when P n = P T, the accelerating force of P is neither increased nor diminished, and the apses are quiescent,

by the triangles $PT : PK :: PM = 3PK : Pn = PT$

$\therefore$ in the required position $3\,PK^2 = PT^2$

or

$$PK = \frac{PT}{\sqrt{3}} = PT \cdot \sin.\theta,$$

$$\therefore \sin. \theta = \frac{1}{\sqrt{3}},$$

or

$$\theta = 35^\circ\ 26'.$$

The addititious force P T — P n is a maximum in quadratures.

$$\text{For } PT : PK :: 3PK : Pn = \frac{3PK^2}{PT}$$

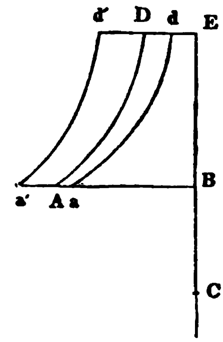
$\therefore PT - Pn = PT - \frac{3PK^2}{PT}$, which is a maximum when $PK = 0$, or the body is in syzygy.

450. Cor. 8. Since the progression or regression of the Apsides depends on the decrement of the force in a greater or less ratio than D^2, from the lower apse to the upper, and on a similar increment from the upper to the lower, (by the IXth Sect.), and is ∴ greatest when the proportion of the force in the upper apse to that in the lower, recedes the most from the inverse square of D, it is manifest that the Apsides progress the fastest from the ablatitious force, when they are in the syzygies, (because the whole forces in conjunction and opposition, i. e. at the upper and lower apses being $\frac{T}{r^2} - \frac{2Sr}{R^3}$, when the apsides are in the syzygies and when r is greatest at the upper apse, $\frac{T}{r^2}$ being least, and the negative part of the expression $\frac{2Sr}{R^3}$ being greatest, the whole expression is ∴ least, and when r is least, at the lower apse, $\frac{T}{r^2}$ being greatest, and the negative part least, ∴ the whole expression is greatest, and ∴ the disproportion between the forces at the upper and lower apse is greatest), and that they regress the slowest in that case from the addititious force, (for $\frac{T}{r^2} + \frac{Sr}{R^3}$, which is the whole force in the quadratures, both before and after conjunction, r being the semi minor axis in each case, differs least from the inverse square); therefore, on the whole the progression in the course of a revolution is greatest when the apsides are in the syzygies.

Similarly the regression is greatest when the apsides are in the quadratures, but still it is not equal to the progression in the course of the revolution.

451. Cor. 8. Let the apsides be in the syzygies, and let the force at the upper apse : that at the lower, : : D E : A B, D A'

being the curve whose ordinate is inversely as the distance[2] from C, $\therefore$ these forces being diminished, the force D E at the upper apse by the greatest quantity $\frac{2 r S}{R^3}$, and the force A B at the lower apse by the least quantity $\frac{2 r' S}{R^3}$; the curve a d which is the new force curve has its ordinates decreasing in a greater ratio than $\frac{1}{D^2}$.



Let the apsides be in the quadratures, then the force E D will be increased by the greatest quantity $\frac{S r}{R^3}$, and the force A B by the least quantity $\frac{S r'}{R^3}$, $\therefore$ the curve a' d' which is the new force curve will have its ordinates decreasing in a less ratio than $\frac{1}{D^2}$.

451. COR. 9. Suppose the line of apsides to be in quadratures, then while the body moves from a higher to a lower apse, it is acted on by a force which does not increase so fast as $\frac{1}{D^2}$ (for the force $= \frac{R^3 T + S r^3}{r^2 R^3}$, $\therefore$ the numerator decreases as the denominator increases), $\therefore$ the orbit will be exterior to the elliptic orbit and the excentricity will be decreased. Also as the descent is caused by the force $\frac{S r}{R^3}(1 - 3 \cos.^2 A)$, the less this force is with respect to $\frac{T}{r^2}$, the less will the excentricity be diminished. Now while the line of the apsides moves from the line of quadratures, the force $\frac{S r}{R^3}(1 - 3 \cos.^2 A)$ is diminished, and when it is inclined at $\angle$ 35° 16′ the disturbing force $= 0$, and $\therefore$ at those four points the excentricity is unaltered. After this, it may be shown in the same manner that the excentricity will be continually increased until the line of apsides coincides with the line of syzygies. Here it is a maximum, since the disturbing force is negative. Afterwards it will decrease as before it increased until the line of apsides again coincides with the quadrature, and then the excentricity = maximum.

(Hence Evection in Astron.)

452. Lemma. To calculate that part of the ablatitious force which is employed in drawing P from the plane of its orbit.

Let A = angular distance from syzygy.

Q = angular distance of nodes from syzygy.

I = inclination of orbit to orbit of S and T.

Then the force required $= \frac{3\,S\,r}{R^3} \cdot \cos. A \cdot \sin. Q \cdot \sin. I.$ (not quite accurately.)

When P is in quadratures, this force vanishes, since cos. A = 0.

When nodes are in syzygy, ——— since sin. Q = 0,

——— quadratures, this force (cæt. par.) = maximum, since sin. Q = sin. 90 = rad.

453. Cor. 12. The effects produced by the disturbing forces are all greater when P is in conjunction than when in opposition.

For they involve $\frac{1}{R^3}$, ∴ when R is least, they are greatest.

454. Cor. 13. Let S be supposed so great that the system P and T revolve round S fixed. Then the disturbing forces will be of the same kind as before, when we supposed S to revolve round T at rest.

The only difference will be in the magnitude of these forces, which will be increased in the same ratio as S is increased.

455. Cor. 14. If we suppose the different systems in which S and S T $\propto$, but P T and P and T remain the same, and the period (p) of P round T remains the same, all the errors $\propto \frac{S}{R^3} \propto \frac{\Delta \cdot d^3}{R^3}$, if Δ = density of S, and d its diameter,

$\propto \delta^3$, if Δ given, and δ=apparent diam.

also

$$\frac{1}{P^2} \propto \frac{S}{R^3} \text{ if } P = \text{period of T round S},$$

$$\therefore \text{the errors} \propto \frac{1}{P^2}.$$

These are the linear errors, and angular errors $\propto$ in the same ratio, since P T is given.

456. Cor. 15. If S and T be varied in the same ratio,

Accelerating force of S : that of T :: $\frac{S}{R^2} : \frac{T}{r^2}$ the same ratio as before.

∴ the disturbances remain the same as before.

(The same will hold if R and r be also varied proportionally.)

∴ the linear errors described in P's orbit $\propto$ P T, (since they involve r), if P T $\propto$, the rest remaining constant.

also the angular errors of P as seen from T $\propto \frac{\text{linear errors}}{\text{P T}} \propto \frac{\text{P T}}{\text{P T}} \propto 1$, and are ∴ the same in the two systems.

The similar linear errors $\propto$ f. T², ∴ P T $\propto$ f. T², and f $\propto \frac{\text{P T}}{\text{T}^2}$, but f $\propto$ accelerating force of T on P $\propto \frac{\text{P T}}{\text{p}^2}$, (p = period of P round T,)

$$\therefore \text{T} \propto \text{p and} \therefore \propto \text{P}$$

$$(\text{for } \text{P}^2 \propto \frac{\text{S T}^3}{\text{S}} \propto \frac{\text{P T}^3}{\text{T}} \propto \text{p}^2)$$

COR. 14. In the systems

S, T, P, Radii R, r Periods P, p
S′, T, P —— R′, r ——— P′, p.

Linear errors dato t. in 1st. : do. in second :: $\frac{1}{\text{P}^2} : \frac{1}{\text{P}'^2}$

∴ angular errors in the period of P —— : —— :: $\frac{1}{\text{P}^2} : \frac{1}{\text{P}'^2}$.

COR. 15. In the systems

S, T, P, —— R, r ——— P, p
S′, T′, P —— R′, r′ ——— P′, p′,

so that $\frac{\text{S}'}{\text{S}} = \frac{\text{T}'}{\text{T}}$ and $\frac{\text{R}'}{\text{R}} = \frac{\text{r}'}{\text{r}}$

$$\therefore \frac{\text{P}}{\text{P}'} = \frac{\text{p}}{\text{p}'}.$$

Linear errors in a revolution of P in 1st. : do. in second :: r : r′
angular errors : :: 1 : 1.

COR. 16. In the systems

S, T, P, —— R, r ——— P, p
S, T′, P, —— R, r′ ——— P, p′.

Linear errors in a revolution of P in 1st. : do in second :: r p² : r′ p′²
angular errors in a revolution of P : :: p² : p′².

To compare the systems

(1) S, T, P —— R, r ——— P, p
(2) S′, T′, P′ —— R′, r′ ——— P′, p′.

Assume the system

(3) S′, T, P —— R′, r ——— P′, p

∴ by (14) angular errors in P S revolution in (1) : in (3) :: $\frac{1}{\text{P}^2} : \frac{1}{\text{P}'^2}$

by (16) angular errors in (3) : in (2) :: p² : p′²

therefore errors in (1) : in (2) :: $\frac{\text{p}^2}{\text{P}^2} : \frac{\text{p}'^2}{\text{P}'^2}$.

Or assume the system (3) Σ, T, P — ϱ, r — Π, p

$$\text{so that } \frac{\Sigma}{S'} = \frac{T}{T'}, \frac{Q}{R'} = \frac{r}{r'}$$

$$\therefore \text{the errors in (1) : errors in (3)} :: \frac{1}{P^2} : \frac{1}{\Pi^2} :: \frac{S}{R^3} : \frac{\Sigma}{\varrho^3} :: \frac{S}{\Sigma} : \frac{R^3}{\varrho^3}$$

$$(3) : \quad (2) :: 1 : 1$$

$$:: \frac{S}{S'}\frac{S'}{\Sigma} : \frac{R^3}{R'^3} \cdot \frac{R'^3}{\varrho^3} :: \frac{S}{S'} \cdot \frac{T'}{T} : \frac{R^3}{R'^3} \cdot \frac{r'^3}{r^3}$$

$$:: \frac{S}{R^3}\frac{r^3}{T} : \frac{S'}{R'^3} \cdot \frac{r'^3}{T} :: \frac{p^2}{P^2} : \frac{p'^2}{P'^2}.$$

457. Cor. 16. In the different systems the mean angular errors of P $\propto \frac{p}{P^2}$ whether we consider the motion of apses or of nodes (or errors in latitude and longitude.)

For first, suppose every thing in the two different systems to be the same except P T, $\therefore$ p will vary. Divide the whole times p, p', into the same number of indefinitely small portions proportional to the wholes. Then if the position of P be given, the disturbing forces all $\propto$ each other $\propto$ P T; and the space $\propto$ f. T^2, $\therefore$ the linear errors generated in any two corresponding portions of time $\propto$ P T. p^2.

$\therefore$ the angular errors generated in these portions, as seen from T, $\propto$ p^2.

$\therefore$ Comp^o. the periodic angular errors as seen from T $\propto$ p^2.

Now by Cor. 14, if in two different systems P T and $\therefore$ p be the same, every thing else varying, the angular errors generated in a given time, as in p, $\propto \frac{1}{P^2}$.

$\therefore$ neutris datis, in different systems the angular errors generated in the time p $\propto \frac{p^2}{P^2}$.

Now

$$p'' : 1'' :: \frac{p^2}{P^2} : \frac{p}{P^2},$$

$\therefore$ the angular errors generated in 1'' (or the mean angular errors) $\propto \frac{p}{P^2}$.

Hence the mean motion of the nodes as seen from T $\propto$ mean motion of the apses, for each $\propto \frac{p}{P^2}$.

458. Cor. 17.

Mean addititious force : mean force of P on T :: p^2 : P^2.

For

mean addititious force : force of S on T :: P T : S T,

$$\left(:: \frac{S\,r}{R^3} : \frac{S}{R^2} :: r : R\right)$$

$$\text{force of S on T : mean force of T on P} :: \frac{S\,T}{p^2} : \frac{P\,T}{p^2}$$

$$\left(\text{force} \propto \frac{\text{rad.}}{p^2}\right)$$

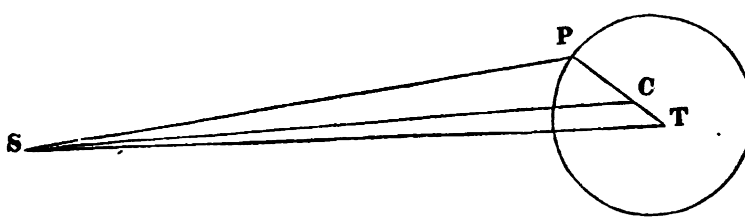
$\therefore$ mean addititious force : mean force of T on P :: $p^2 : P^2$

$\therefore$ ablatitious force : mean force of T on P :: $3 \cos. \theta . p^2 : P$.

Similarly, the tangential and central ablatitious and all the forces may be found in terms of the mean force of T on P.

459. PROP. LXVII. Things being as in Prop. LXVI, S describes the areas more nearly proportional to the times, and the orbit more elliptical round the center of gravity of P and T than round T.

For the forces on S are $\frac{P}{PS^2}$ and $\frac{T}{TS^2}$.



$\therefore$ the direction of the compound force lies between S P, S T; and T attracts S more than P.

$\therefore$ it lies nearer T than P, and $\therefore$ nearer C the center of gravity of T and P.

$\therefore$ the areas round C are more proportional to the times, than when round T.

Also as S P increases or decreases, S C increases or decreases, but S T remains the same; $\therefore$ the compound force is more nearly proportional to the inverse square of S C than of S T; $\therefore$ also the orbit round C is more nearly elliptic (having C in the focus) than the orbit round T.

A

SECOND COMMENTARY

ON

SECTION XI.

---

460. To find the axis major of an ellipse, whose periodic time round S at rest would equal the periodic time of P round S in motion.

Let A equal the axis major of an ellipse described round P at rest equal the axis major of P Q v.

Let x equal the axis major required,

P. T. of P round S in motion : p S at rest : : $\sqrt{S} : \sqrt{S+P}$

P. T. of p in the elliptic axis A : P. T. in the elliptic axis x : : $A^{\frac{3}{2}} : x^{\frac{3}{2}}$

$\therefore$ P. T. of P round S in motion : P.T. in the el.ax. x : : $\sqrt{A^3 S} : \sqrt{x^3 (S+P)}$.

By hyp. the 1st term equals the 2d,

$$\therefore A^3 S = x^3 . \overline{S+P}$$

$$\therefore A : x :: (S+P)^{\frac{1}{3}} : S^{\frac{1}{3}}.$$

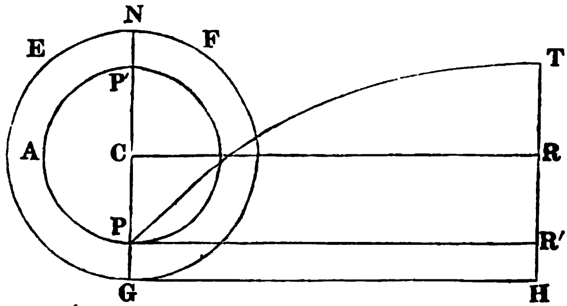
461. Prop. LXIII. Having given the velocity, places, and directions of two bodies attracted to their common center of gravity, the forces varying inversely as the distance^2, to determine the actual motions of bodies in fixed space.

Since the initial motions of the bodies are given, the motions of the center of gravity are given. And the bodies describe the same moveable curve round the center of gravity as if the center were at rest, while the center moves uniformly in a right line.

* Take therefore the motion of the center proportional to the time, i. e. proportional to the area described in moveable orbits.

* Since a body describes some curve in fixed space, it describes areas in proportion to the times in this curve, and since the center moves uniformly forward, the space described by it is in proportion to the time, therefore, &c.

462. Ex. 1. Let the body P describe a circle round C, while the center C moves uniformly forward. Take C G : C P : : v of C : v of P, and with the



center C and rad. C G describe a circle G C N, and suppose it to move round along G H, then P will describe the trochoid P L T, and when P has described the semicircle P A B, P will be at the summit of the trochoid

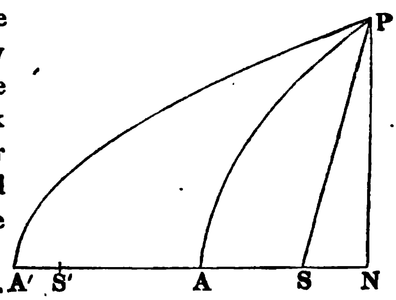
∴ every point of the semicircumference G F N will have touched G H,

∴ G H equals the semicircumference G F N,

∴ v of P : v of C : : P A B semicircumference : C R = G F N semicircle

* : : C P : C G Q. e. d.

463. Ex. 2. Let the moveable curve be a parabola, and let the center of gravity move in the direction of its primitive axis. When the body is at the vertex A′, let S′ be the position of the center of gravity, and while S′ has described uniformly S′ S, let A have described the arc of the parabola A P.



Let A′ N = x, N P = y, be the abscissa and ordinate of the curve A P in fixed space.

Let 4 p equal the parameter of the parabola A P.

$$\therefore A N = \frac{y^2}{4p}, \quad A' S = S' S = x - \frac{y^2}{4p} = \frac{4px - y^2}{4p}$$

$$S N = A N - A S = A N - p = \frac{y^2}{4p} - p' = \frac{y^2 - 4p^2}{4p}$$

$$\text{Area } A S P = A N P - S N P = \tfrac{2}{3} A N \times N P - \tfrac{1}{2} N S \times N P$$

$$= \tfrac{2}{3} \cdot \frac{y^3}{4p} - \tfrac{1}{2} \frac{y^3 - 4p^2 y}{4p} = \frac{y^3 + 12p^2 y}{24p}.$$

By Prop. S′ S ∝ A S P; therefore they are in some given ratio.

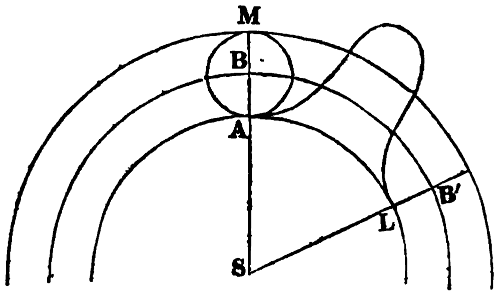
$$\text{Let } A S P : S' S : : a : b : : \frac{y^3 + 12p^2 y}{24p} : \frac{4px - y^2}{4p}$$

* If C P = C G the curve in fixed space becomes the common cycloid.
If C P > C G ——————————— the oblongated trochoid.

$$\therefore y^3 + 12\,p^2\,y = 4\,p\,a\,x - a\,y^2$$
$$\therefore y^3 + a\,y^2 + 12\,p^2\,y - 4\,p\,a\,x = 0.$$

Equation to the curve in fixed space.

464. Ex. 3. * Let B B′ be the orbit of the earth round the sun, M A

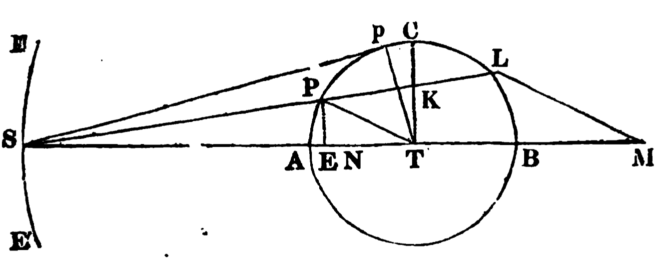


that of the moon round the earth, then the moon will, during a revolution, trace out a contracted or protracted epicycloid according as A L has a greater or less circumference than A M, and the orbit of the moon round the sun will consist of twelve epicycloids, and it will be always concave to the sun. For

$$\text{F of the earth to the sun : F of the moon to the earth} :: \frac{R}{P^2} : \frac{r}{p^2}$$
$$:: \frac{400}{(365)^2} : \frac{1}{(27)^2}$$

in a greater ratio than 2 : 1. But the force of the earth to the sun is nearly equal to the force of the moon to the sun, $\therefore$ the force of the moon to the earth, $\therefore$ the deflection to the sun will always be within the tangential or the curve is always concave towards the sun.

465. Prop. LXVI. If three bodies attract each other with forces



varying inversely as the square of the distance, but the two least revolve

* To determine the nature of the curve described by the moon with respect to the sun.

about the greatest, the innermost of the two will more nearly describe the areas proportional to the time, and a figure more nearly similar to an ellipse, if the greatest body be attracted by the others, than if it were at rest, or than if it were attracted much more or much less than the other bodies.

$$(\mathrm{LM} : \mathrm{PT} :: \mathrm{SL} : \mathrm{SP},$$

$$\therefore \mathrm{LM} \propto \frac{\mathrm{PT}}{\mathrm{SP}^3},$$

$$\therefore \mathrm{LM} = \frac{\mathrm{PT} \times \mathrm{SL}}{\mathrm{SP}} = \frac{\mathrm{SK}^2 \times \mathrm{PT}}{\mathrm{SP}^3}\Big),$$

$$\therefore \mathrm{SK}^2 : \mathrm{SP}^2 :: \mathrm{SL} : \mathrm{SP}).$$

Let P and S revolve in the same plane about the greatest body T, and P describe the orbit P A B, and S, E S E. Take S K the mean distance of P from S, and let S K represent the attraction of P to S at that distance. Take S L : S K :: S R^2 : S P^2, and S L will represent the attraction of S on P at the distance S P. Resolve it into two S M, and L M parallel to P T, and P will be acted upon by three forces P T, L M, S M. The first force P T tends to T′, and varies inversely as the distance^2, $\therefore$ P ought by this force to describe an ellipse, whose focus is T. The second, L M, being parallel to P T may be made to coincide with it in this direction, and $\therefore$ the body P will still, being acted upon by a centripetal force to T, describe areas proportional to the time. But since L M does not vary inversely as P T, it will make P describe a curve different from an ellipse, and $\therefore$ the longer L M is compared with P T, the more will the curves differ from an ellipse. The third force S M, being neither in the direction P T, nor varying in the inverse square of the distance, will make the body no longer describe areas in proportion to the times, and the curve differ more from the form of an ellipse. The body P will $\therefore$ describe areas most nearly proportional to the times, when this third force is a minimum, and P A B will approach nearest to the form of an ellipse, when both second and third forces are minima. Now let S N represent the attraction of S on T towards S, and if S N and S M were equal, P and T being equally attracted in parallel directions would have relatively the same situation, and if S N be greater or less then S M, their difference M N is the disturbing force, and the body P will approach most nearly the equable description of areas, and P A B to the form of an ellipse, when M N is either nothing or a minimum.

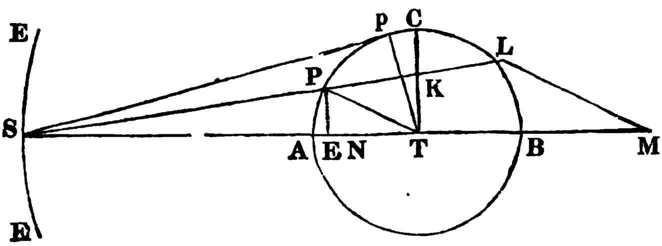
Case 2. If the bodies P and S revolve about T in different planes, L M being parallel to P S will have the same effect as before, and will not

1

tend to move P from its plane. But N M acting in a different plane, will tend to draw P out of its plane, besides disturbing the equable description of areas, &c. and as before this disturbing force is a minimum, when M N is a minimum, or when S N = nearly S K.

466. To estimate the magnitude of the disturbing forces on P, when P moves in a circular orbit, and in the same plane with S and T.

Let the angle from the quadratures P C T = θ,



$S\,T = d$, $P\,T = r$, F at the distance (a) = M,

$$F \text{ on } P = \frac{M\,a^2}{S\,P^2},$$

$\therefore$ From P in the direction S P : $-$ P T : : S P : P T,

$$\therefore F \text{ in the direction } P\,T = \frac{M\,a^2}{S\,P^2} \times \frac{P\,T}{S\,P}.$$

But $S\,P^2 = d^2 + r^2 - 2\,d\,r \sin.\theta$,

$$\therefore F \text{ in the direction } P\,T = \frac{M\,a^2\,r}{(d^2 + r^2 - 2\,d\,r \sin.\theta)^{\frac{3}{2}}}$$

$$= \frac{M\,a^2\,r}{d^3}\left\{1 - \frac{3}{2}\,\frac{r^2 - 2\,d\,r \sin.\theta}{d^2}\right\}$$

$= \frac{M\,a^2\,r}{d^3} = A$ nearly, since d being indefinitely great compared with r in the expansion, all the terms may be neglected except two. First $\frac{1}{d}$ vanishes when compared with $\frac{1}{d^3}$, $\therefore$ the addititious force in the direction F T = A. By proportion as before, force in the direction S T

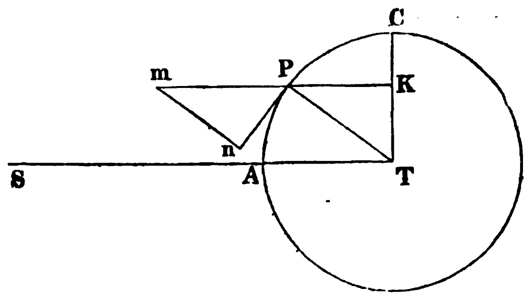
$$= \frac{M\,a^2}{S\,P^2}\,\frac{S\,T}{S\,P} = \frac{M\,a^2\,d}{d^3\left(1 + \frac{r^2 - 2\,d\,r \sin.\theta}{d^2}\right)^{\frac{3}{2}}}$$

$$= \frac{M\,a^2}{d^2}\left(1 - \frac{3}{2}\,\frac{r^2 - 2\,d\,r \sin.\theta}{d^2}\right)$$

$$= \frac{M\,a^2}{d^2} - \frac{3\,M\,a^2\,r^2}{2\,d^4} + \frac{3\,M\,a^2\,r \sin.\theta}{d^3},$$

$\therefore$ force in the direction S T $= \dfrac{\text{M a}^2}{\text{d}^2} + \dfrac{3 \text{ M a}^2 \text{r}}{\text{d}^3}$ sin. θ nearly, since $\dfrac{1}{\text{d}^4}$ vanishes when compared with $\dfrac{1}{\text{d}^3}$, and the force of S on T $= \dfrac{\text{M a}^2}{\text{d}^2}$,

$$\therefore \text{ ablatitious F} = \frac{\text{M a}^2}{\text{d}^2} + \frac{3 \text{ M a r}}{\text{d}^3} . \text{sin. } \theta - \frac{\text{M a}^2}{\text{d}^2}$$
$$= 3 \text{ A} . \text{sin. } \theta.$$



If P T equal the addititious force, then the ablatitious force equals 3 P K, for P K : P T : : sin. θ : (1 = r),

$$\therefore 3 \text{ P K} = 3 \text{ P T} . \text{sin. } \theta = 3 \text{ A} . \text{sin. } \theta.$$

To resolve the ablatitious force. Take

$$\text{P m} : \text{P n} : : \text{P T} : \text{T K} : : 1 : \text{cos. } \theta,$$
$$\therefore \text{P n} = \text{P m} \times \text{cos. } \theta = 3 \text{ A} \times \text{sin. } \theta \text{ cos. } \theta = \frac{3 \text{ A}}{2} . \text{sin. } 2\,\theta .$$
$$\text{m n} = \text{P m} \times \text{P K} = 3 \text{ A} . \text{sin.}^2 \theta = 3 \text{ A} . \frac{1 - \text{cos. } 2\,\theta}{2},$$

$\therefore$ the disturbing forces of S on P are

1. The addititious force $= \dfrac{\text{M a}^2 \text{r}}{\text{d}^3} = \text{A}$.

2. The ablatitious force which is resolved into the tangential part $= \dfrac{3 \text{ A}}{2} . \text{sin. } 2\,\theta$, and that in the direction T P $= 3 \text{ A} . \dfrac{1 - 2 \text{ cos. A}}{2}$,

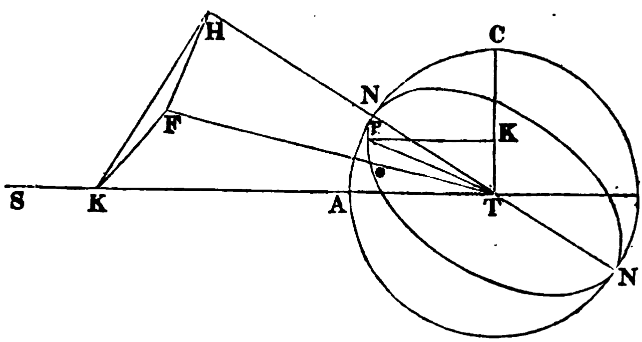
$\therefore$ whole disturbing force in the direction P T $= \text{A} - 3 \text{ A} . \dfrac{1 - 2 \text{ cos. A}}{2}$

$= \text{A} - \dfrac{3 \text{ A}}{2} + \dfrac{3 \text{ A}}{2} . \text{cos. } 2\,\theta = -\dfrac{\text{A}}{2} + \dfrac{3 \text{ A}}{2} . \text{cos. } 2\,\theta$, and in the whole revolution the positive cosine destroys the negative, therefore the whole disturbing force in a complete revolution is ablatitious, and equal to one half of the mean addititious force.

467. To compare N M and L M.

$$\text{L M} : \text{P T} : : \left(\text{S L} = \frac{\text{S K}^3}{\text{S P}^2}\right) : \text{S P},$$
$$\therefore \text{L M} = \frac{\text{S K}^3}{\text{S P}^3} \times \text{P T}$$

$$MN = \frac{SK^3}{SP^3} \times ST - ST = \frac{SK^3 - SP^3}{SP^3} \times ST$$
$$= \frac{SK^3 - (SK - KP)^3}{SP^3} \times ST$$
$$= \frac{SK^3 - SK^3 + 3SR^2 \times KP}{SP^3} \times ST \text{ nearly}$$
$$= \frac{3SK^2 \times PK}{SP^3} \times ST \text{ nearly} = \frac{3SK^3}{SP^3} \times PK$$
$$= \frac{3SK^3}{SP^3} \times PT \times \sin.\theta,$$
$$\therefore MN : LM :: 1 : 3 \sin.\theta.$$



468. Next let S and P revolve about T in different planes, and let N P N′ be P's orbit, N N′ the line of the nodes. Take T K in T S = 3 A . sin. θ. Pass a plane through T K and turn it round till it is perpendicular to P's orbit. Let T e be the intersection of it with P's orbit. Produce T E and draw K F perpendicular to it, ∴ K F is perpendicular to the plane of P's orbit, and therefore perpendicular to every line meeting it in that orbit, T in the plane of S's orbit; draw K H perpendicular to N′ N produced; join H F, then F H K equals the inclination of the planes of the two orbits. For K H T, K F T, K F H being all right angles,

$$KT^2 = KH^2 + HT^2$$
$$KF^2 + H^2 = KF^2 + FH^2 + HT^2,$$
$$^* \therefore FT^2 = FH^2 + HT^2,$$
$$\therefore FH \text{ is perpendicular to } HT.$$

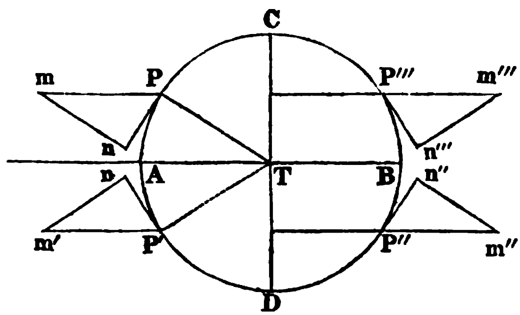
Since $PT = A$, $TK = A \times \sin.\theta$

* Let the angle K H T = T, H T K = ϕ = angular distance of the line of the nodes from S y z.

$$PT : TK :: 1 : 3 \sin.\theta$$
$$TK : KH :: 1 : \sin.\varphi$$
$$KH : KF :: 1 : \sin. T,$$
$$\therefore PT : KF :: 1 : 3 \sin.\theta.\sin.\varphi.\sin. T,$$

$\therefore$ ablatitious force perpendicular to P's orbit $= KF$
$= 3\,PT \times \sin.\theta.\sin.\varphi \times \sin. T = 3\,A \times \sin.\theta.\sin.\varphi \times \sin. T.$

2d. Hence it appears that there are four forces acting on P.



1. Attraction of P to T $\propto \frac{1}{r^2}$.

2. Addititious F in the direction $PT = \frac{M a^2 r}{d^3}$.

3. Ablatitious F in the direction $PT = \frac{3 M a^2 r}{d^3} . \sin.^2 \theta$.

4. Tangential part of the ablatitious force $= \frac{3}{2} . \frac{M a^2 r}{d^3} . \sin.^2 \theta$.

Of these the three first acting in the direction of the radius-vector do not disturb the equable description of areas, the fourth acting in the direction of a tangent at P does interrupt it.

Since the tangential part of F is formed by the revolution of $PM = 3\,A \times \sin.\theta$ at C, $\theta = 0$, therefore $Pm = 0$, and consequently the tangential $F = 0$; from C to A, P n is in consequentia, and therefore accelerates the body P at A, it again equals 0, and from A to D is in antecedentia, and therefore retards P; from D to B it accelerates; from B to C it retards.

Therefore the velocity of P is greatest at A and B, because these are the points at which the accelerations cease and retardations begin, and the velocity is least at D and C. To find the velocity gained by the action of the tangential force.*

$$dZ = F\,dx = \tfrac{3}{2}\,A.\sin. 2\theta\, d\theta$$

* F in the direction P T is a maximum at the quadrature, because the ablatitious F in the quadrature is 0, and at every other point it is something.

$\sin. 2\,\theta \times 2\,\theta' = -(\cos. 2\,\theta)'$,

$$\therefore Z = \frac{v^2}{2\,g} = \text{Cor.} - \tfrac{3}{4}\,A.\cos. 2\,\theta.$$

But when $\theta = 0$, the tangential $F = 0$, and no velocity is produced,

$\therefore \cos. 2\,\theta = R = 1$,

$$\therefore \frac{v^2}{2\,g} = \frac{3\,A}{4}(1 - \cos. 2\,\theta) = \tfrac{3}{4}\,A.\,2\sin.^2\theta,$$

$$\therefore v^2 = 3\,g\,A.\sin.^2\theta,$$

$$\therefore v = \sqrt{3\,g\,A.}\,\sin.\theta,$$

$$\therefore v' \propto (\sin.\theta)',$$

$\therefore$ whole f on the moon at the mean distance : f of S on T :: $\frac{1}{p^2} : \frac{d}{P^4}$

and the force of S on T : add. f at the mean distance (m) :: $\frac{q}{d^2} : \frac{q}{d^3}$,

$\therefore$ whole f at the mean distance : m :: $P^2 : p^2$ and $\frac{p^2}{P^2} \times$ whole f &c. = m.

Now f on the moon at any distance (r) $= \frac{f}{r^2} - \frac{q\,r}{2\,d^3}$ and at the mean distance (1) $= f - \frac{q}{2\,d^3} = f - \frac{m}{2}$,

$$\therefore m = \frac{p^2 f}{P^2} - \frac{m\,p^2}{2\,P^2},$$

$$\therefore m = \frac{2\,p^2 f}{2\,P^2 + p^2},$$

and therefore nearly $= \frac{p^2 f}{\rho} - \frac{p^4 f}{2\,P^4}$,

$\therefore$ m r (which equals the addititious force) $= f.\,r.\left\{\frac{p^2}{P^2} - \frac{2\,p^4}{P^4}\right\}$.

469. To compare the ablatitious and addititious forces upon the moon, with the force of gravity upon the earth's surface. (Newton, Vol. III. Prop. XXV.)

add. f : f of S on T :: P T : S T

f of S on T : f of the earth on the moon :: $\frac{S\,T}{P^3} : \frac{P\,V}{p^2} = \frac{P\,T}{p^3}$,

$\therefore$ add. f : f of the earth on the moon :: $p^3 : P^3$

f of the earth on the moon : force of gravity :: 1 : 60^2,

$\therefore$ add. f : force of gravity :: $p^3 : P^3.\,60^2$ . . . (1)

Also ablat. f : addititious force :: 3 P K : P T,

$\therefore$ ablat. f : addititious force :: $3\,P\,K.\,p^3 : 60^2.\,P\,T.\,P^3$. (2)

470. Cor. 2. In a system of three bodies S, P, T, force $\propto \frac{1}{d^2}$, the

body P will describe greater areas in a given time at the syzygies than at the quadrature.

The tangent ablatitious $f = \frac{3}{2} . PT . \sin. 2\theta$; therefore this force will accelerate the description of areas from quadratures to syzygies and retard it from syzygies to quadratures, since in the former case sin. 2θ is positive, and in the latter negative.

COR. 3. is contained in Cor. 2.

The first quadrant d. sin. being positive the velocity increases, in the second d. sin. negative the velocity decreases, &c. for the 1st Cor. 2d Cor. &c.

Also v is a maximum when sin. θ is a maximum, i. e. at A and B.

471. COR. 4. The curvature of P's orbit is greater in quadratures than in the syzygy.

$$\text{The whole F on P} = \frac{Ma^2}{r^2} + \frac{Ma^2r}{d^3} - \frac{3Ma^2r}{2d^3}(1 - \cos. 2\theta) \times \left(\frac{3Ma^2r . \sin. 2\theta}{2d^3}\right).$$

In quadratures sin. $2\theta = 0$,

$$\therefore F = \frac{Ma^2}{r^2} + \frac{Ma^2r}{d^3}.$$

And in syz. $2\theta = 180$,

$$\therefore \sin. 2\theta = 0, \quad \cos. 2\theta = 1$$

$$\therefore \frac{3Ma^2r}{2d^3} . (1 - \cos. 2\theta) = \frac{3Ma^2r}{d^3},$$

$$\therefore \text{the whole F on P in the syz.} = \frac{Ma^2}{r^2} - \frac{2Ma^2r}{d}$$

$\therefore$ F is greater in the quadratures than in the syzygies; and the velocity is greater in the syzygies than in the quadratures.

But the curvature $\propto \frac{1}{PV} \propto \frac{F}{V^2}$, $\therefore$ is greatest in the quadratures and least in the syzygies.

472. COR. 5. Since the curvature of P's orbit is greatest in the quadrature and least in the syzygy, the circular orbit must assume the form of an ellipse whose major axis is C D and minor A B.

$\therefore$ P recedes farther from T in the quadrature than in the syzygy.

473. COR. 6.

$$\text{The whole F on P in the line PT} = \frac{Ma^2}{r^2} + \frac{Ma^2r}{d^3} - \frac{3Ma^2r}{d^3} . \sin.^2\theta$$

$$= \text{in quad.} \frac{Ma^2}{r^2} + \frac{Ma^2r}{d^3}$$

$$\text{and in syz.} = \frac{M a^2}{r^2} - \frac{2 M a^2 r}{d^3}$$

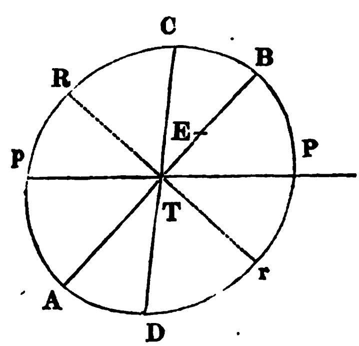
let the ablatitious force on P equal the addititious, and

$$\frac{M a^2 r}{d^3} = \frac{3 M a^2 r}{d^3} . \sin.^2 \theta$$

$$\therefore \sin. \theta = \frac{1}{\sqrt{3}} \sin. 35° . 16.$$

Therefore up to this point from quadrature the ablatitious force is less than the addititious, and from this to one equally distant from the other point of quadrature, the ablatitious is greater than the addititious, therefore in a whole revolution the gravity of P to T is diminutive from what it would be if the orbit were circular or if S did not act, and P $\propto \frac{R^{\frac{3}{2}}}{\sqrt{\text{abl. F}}}$ and since the action of S is alternately increased or diminished, therefore P $\propto$ from what it would be were P T constant, both on account of the variation, and of the absolute force.

474. COR. 7. * Let P revolve round T in an elliptic orbit, the force on P in the quad. $= \frac{M a^2}{r^2} + \frac{M a^2 r}{d^3} + \frac{b}{r^2} + c\, r.$



$\therefore G + 180 \sqrt{\frac{b + c}{b + 4 c}}$ and since the number is greater than the denomination G is less than 180. $\therefore$ the apsides are regressive if the same effect is produced as long as the addititious force is greater than the ablatitious, i. e. through 35°. 16′.

The force on P in the syz. $= \frac{M a^2}{r} - \frac{2 M a^2 r}{d^3} = \frac{b}{r^2} - 2 c\, r$

* Since P $\propto \frac{R^{\frac{3}{2}}}{\sqrt{\text{ablatitious force}}}$ and in winter the sun is nearer the earth than in summer, R is increased in winter, and A is diminished, therefore the lunar months are shorter in winter than in summer.

$$\therefore G = 180 \cdot \sqrt{\frac{b - 2c}{b - 8c}} > 180^\circ$$

$\therefore$ in the syz. the apsides are progressive, and since $\sqrt{\frac{b - mc}{b - nc}}$ will be an improper fraction as long as the ablatitious force is greater than the addititious, and when the disturbing forces are equal, $mc = nc$, therefore $G = 180^\circ$, i. e. the line of apsides is at rest (or it lies in V C produced 9th.) $\therefore$ since they are regressive through $141^\circ. 4'$ and progressive $218^\circ. 56'$ they are on the whole progressive.

To find the effect produced by the tangential ablatitious force, on the velocity of P in its orbit. Assume u = velocity of a body at the mean distance 1, then $\frac{u}{r}$ = velocity at any other distance r nearly, the orbit being nearly circular.

Let v be the true velocity of P at any distance (r), $v\,dv = g\,F\,dx$ $\left(\frac{g}{2} = 16\frac{1}{12}\right.$. For the tangent ablatitious $f = \frac{3}{2} . PT . 2\theta$, and $\left.x' = r\,\theta'\right)$ $= 3\,PT . m\,r . \sin. 2\,\theta . \theta'$,

$$\therefore v^2 = -3\,PT\,m\,r\cos. 2\,\theta + C,$$

and

$$C = \frac{u^2}{r^2},$$

$$\therefore v^2 = \frac{u^2}{r^2} - \&c.$$

Hence it appears that the velocity is greatest in syzygy and least in quadrature, since in the former case, cos. $2\,\theta$ is greatest and negative, and in the latter, greatest and positive.

To find the increment of the moon's velocity by the tangential force while she moves from quadrature to syzygy.

$$v^2 = -3\,PT . m . r . \cos. 2\,\theta + C,$$

but (v) the increment $= 0$, when $\theta = 0$,

$$\therefore C = 3\,PT . m . r,$$

$$\therefore v^2 = 3\,PT . m . r\,(1 - \cos. 2\,\theta) = 6\,PT . m . r . \sin.^2\theta,$$

and when $\theta = 90^\circ$, or the body is in syzygy $v^2 = 6\,PT\,m . r$.

475. COR. 6. Since the gravity of P to T is twice as much diminished in syzygy as it is increased in quadrature, by the action of the disturbing force S, the gravity of P to T during a whole revolution is diminished. Now the disturbing forces depend on the proportion between P T and T S, and therefore they become less or greater as T S becomes greater

or less. If therefore T approach S, the gravity of P to T will be still more diminished, and therefore P T will be the increment.

Now $P.T \propto \frac{R^{\frac{3}{2}}}{\sqrt{\text{absolute force}}}$; since, therefore, when S T is diminished, R is increased and the absolute force diminished (for the absolute force to T is diminished by the increase of the disturbing force) the P.T is increased. In the same way when S T is increased the P.T is diminished, therefore P.T is increased or diminished according as S T is diminished or increased. Hence per. t of the moon is shorter in winter than in summer.

OTHERWISE.

476. Cor. 7. To find the effect of the disturbing force on the motion of the apsides of P's orbit during a whole revolution.

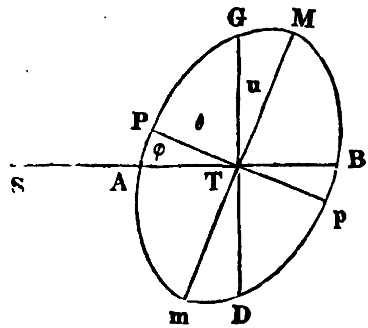
Let f = gravity of P to T at the mean distance (1), then $\frac{f}{r^2}$ = gravity of P at any other distance r.

Now in quadrature the whole force of P to T $= \frac{f}{r^2}$ + add. f $= \frac{f}{r^2} + r = \frac{f\,r + r^4}{r^3}$ and with this force the distance of the apsides $= 180^\circ \sqrt{\frac{f+1}{f+4}}$ which is less than 180°, therefore the apsides are regressive when the body is in quadrature. Now in syz. the whole force of P to T $= \frac{f}{r^2} - 2\,r = \frac{f\,r - 2\,r^4}{r^3}$, therefore the distance between the apsides $= 180^\circ \sqrt{\frac{f-2}{f-8}}$ which is greater than 180°, therefore the apsides are progressive when the body is in syzygy.

But as the force (2 r) which causes the progression in syzygy is double the force (r) which causes the regression in quadrature, the progressive motion in syzygy is greater than the regressive motion in the quadrature. Hence, upon the whole, the motion of the apsides will be progressive during a whole revolution.

At any other point, the motion of the apsides will be progressive or retrograde, according as the whole central force $-\frac{P\,T}{2} + \frac{3\,P\,T}{2}.\cos.2\,\theta$ is negative or positive.

477. COR. 8. To calculate the disturbing force when P's orbit is excentric.



The whole central disturbing force $= -\frac{P T}{2} + \frac{3 P T}{2} \cos. 2\theta =$ $-\frac{m r}{2} + \frac{3 m r}{2} . \cos. 2\theta$ (m is the mean add. f). Now $r = \frac{1 - e^2}{1 - e \cos. u}$ $=$ by div. $1 - e^2 + e . \cos. u + e^2 . \cos.^2 u$, &c. neglecting terms involving e^3, &c. $= 1 - \frac{e^2}{2} + e . \cos. u + \frac{e^2}{2} . \cos. 2u$; therefore the whole central disturbing force $= -\frac{m}{2} + \frac{m e^2}{4} - \frac{m . e . \cos. u}{2} -$ $\frac{m e^2 \cos. 2u}{4} + \frac{3}{2} m \cos. 2\theta - \frac{3 m e^2}{2} . \cos. 2\theta + \frac{3}{2} m e . \cos. u . \cos. 2\theta$ $+ \frac{3}{4} m e^2 . \cos. 2u . \cos. 2\theta$.

478. COR. 8. It has been shown that the apsides are progressive in syzygy in consequence of the ablatitious force, and that they are regressive in quadrature from the effect of the ablatitious force, and also, that they are upon the whole progressive. It follows, therefore, that the greater the excess of the ablatitious over the addititious force, the more will the apsides be progressive in the course of a revolution. Now in any position m M of the line of the apsides, the excess of the ablatitious in conjunction $=$ 2 A T in opposition $=$ T B, therefore the whole excess $=$ 2 A B. Again, the excess of the addititious above the ablatitious force in quadrature $=$ C D. Therefore the apsides in a whole revolution will be retrograde if 2 A B be less than C D, and progressive if 2 A B be greater than C D. Also their progression will be greater, the greater the excess of 2 A B above C D; but the excess is the greatest when M m is in syzygy, for then A B is greatest and C D the least. Also, when M m is in syzygy the apsides being progressive are moving in the same direction with S, and therefore will remain for some length of time in syzygy. Again, when the apsides are in quadrature A B $=$ P p, and C D $=$ M m,

but if the orbit be nearly circular, 2 A B is greater than C D; therefore the apsides are still in a whole revolution progressive, though not so much as in the former case.

In orbits nearly circular it follows from $G = \frac{F}{\sqrt{r}}$ when $F \propto A^{p-3}$, that if the force vary in a greater ratio than the inverse square, the apsides are progressive. If therefore in the inverse square they are stationary,—if in a less ratio they are regressive. Now from quadrature to 35° a force which $\propto$ the distance is added to one varying inversely as the square, therefore the compound varies in a less ratio than the inverse square, therefore the apsides are regressive up to this point. At this point $F \propto \frac{1}{\text{distance}^2}$, therefore they are stationary. From this to 35° from another □ a quantity varying as the distance is subtracted from one varying inversely as the square, therefore the resulting quantity varies in a greater ratio than the inverse square, therefore the apsides are progressive through 218°.

OTHERWISE.

479. Cor. 8. It has been shown that the apsides are progressive in syzygy in consequence of the ablatitious force, and that they are regressive in the quadratures on account of the addititious force, and they are on the whole progressive, because the ablatitious force is on the whole greater than the addititious. ∴ the greater the excess of the ablatitious force above the addititious the more will be the apsides progressive.

In any position of the line A B in conjunction the excess of the ablatitious force above the addititious is 2 P T, in opposition 2 p t. ∴ the whole excess in the syzygies = 2 P p. In the quadratures at C the ablatitious force vanishes. ∴ the excess of the addititious = additious = C T. ∴ the whole addititious in the quadratures = C D.

Now the apsides will, in the whole revolution, be progressive or regressive, according as 2 P p is greater or less than C D, and then the progression will be greatest in that position of the line of the apses when 2 P p — C D is the greatest, i. e. when A B is in the syzygy, for then 2 P p = 2 A B, the greatest line in the ellipse, and C D = R r = ordinate = least through the focus. ∴ 2 P p — C D is a maximum. Also when A B is in the syzygy, the line of apsides being progressive, will move the same way as S. ∴ it will remain in the syzygy longer, and on this account the apsides will be more progressive. But when the apsides are in the quadratures S P = R r and C D = A B, and the orbit being nearly circular, R r nearly equals A B. ∴ 2 P p — C D is positive, and the

apsides are progressive on the whole, though not so much as in the last case; and the apsides being regressive in the quadratures move in the opposite direction to S, $\therefore$ are sooner out of the quadratures, $\therefore$ the regression in the quadrature is less than the progression in the syzygy.

480. COR. 9. LEMMA. If from a quantity which $\propto \frac{1}{A^2}$ any quantity be subtracted which $\propto$ A the remainder will vary in a higher ratio than the inverse square of A, but if to a quantity varying as $\frac{1}{A^2}$ another be added which $\propto$ A, the sum will vary in a lower ratio than $\frac{1}{A^2}$.

If $\frac{1}{A^2}$ be diminished C A $= \frac{1 - c A^3}{A^2}$. If A increases $1 - c A^3$ decreases, and $\frac{1}{A^2}$ increases. $\therefore$ the quantity decreases, $1 - c A$ increases and $\frac{1}{A^2}$ increases. $\therefore$ increases from both these accounts. $\therefore$ the whole quantity varies in a higher ratio than $\frac{1}{A^2}$.

If C A be added $\frac{1 + c A^3}{A^2}$, as A is increased the numerator increases, and $\frac{1}{A^2}$ decreases. $\therefore$ the quantity does not decrease so fast as $\frac{1}{A^2}$, and if A be diminished $1 + c A^3$ is diminished, and $\frac{1}{A^2}$ increased. $\therefore$ the quantity is not increased as fast as $\frac{1}{A^2}$. $\therefore$ &c. Q. e. d.

OTHERWISE.

481. COR. 9. To find the effect of the disturbing force on the excentricity of P's orbit. If P were acted on by a force $\propto \frac{1}{d^2}$, the excentricity of its orbit would not be altered. But since P is acted on by a force varying partly as $\frac{1}{d^2}$ and partly as the distance, the excentricity will continually vary.

Suppose the line of the apsides to coincide with the quadrature, then while the body moves from the higher to the lower apse, it is acted upon by a force which does not increase so fast as $\frac{1}{d^2}$, for the force at the quadrature $= \frac{f}{r^2} + m r$, and $\therefore$ the body will describe an orbit exterior to the elliptic which would be described by the force $\propto \frac{1}{d^2}$. Hence the body

will be farther from the focus at the lower apse than it would have been had it moved in an elliptic orbit, or the excentricity is diminished. Also as the decrease in excentricity is caused by the force (m r), the less this force is with respect to $\frac{f}{r^2}$, the less will be the diminution of excentricity. Now while the line of apsides moves from the line of quadratures, the force (m r) is diminished, and when it is inclined at an angle of 35° 16′ the disturbing force is nothing, and $\therefore$ at those four points the excentricity remains unaltered. After this it may be shown in the same manner that the excentricity will be continually increased, until the line of apsides coincides with the syzygies. Hence it is a maximum, since the disturbing force in these is negative. Afterwards it will decrease as before it increased, until the line of apsides again coincides with the line of quadrature, and the excentricity is a minimum.

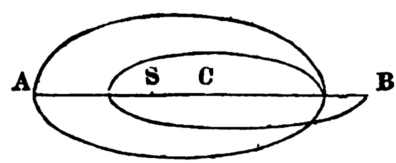
Cor. 14. Let P T = r, S T = d, f = force of T on P at the distance 1, g = force of S on T at the distance, then the ablatitious force $= \frac{3 g r \sin. \theta}{d^3}$; if $\therefore$ the position of P be given, and d varies, the ablatitious force $\propto \frac{1}{d^3}$. But when the position of P is given, the ablatitious : addititious : : in a given ratio, $\therefore$ addititious force $\propto \frac{1}{d^3}$, or the disturbing force $\propto \frac{1}{d^3}$. Hence if the absolute force of S should $\propto$ the disturbing force $\propto \frac{\text{absol. f.}}{d^3}$. Let P = the periodical time of T about S, $\therefore \frac{1}{P^2} \propto \frac{\text{absol. f.}}{d^3}$. Let Δ = density, δ = diameter of the sun, then the absolute force $\propto \Delta\, \delta^3$, then the disturbing force $\propto \frac{\Delta\, \delta^3}{d^3} \propto \frac{1}{P^2} \propto \Delta$ (apparent diameter)^3 of the sun. Or since P T is constant, the linear as well as the angular errors $\propto$ in the same ratio.

483. Cor. 15. If the bodies S and T either remain unchanged, or their absolute forces are changed in any given ratio, and the magnitude of the orbits described by S and P be so changed that they remain similar to what they were before, and their inclination be unaltered, since the accelerating force of P to T : accelerating force of S : : $\frac{\text{absolute force of T}}{\text{P T}^2}$: $\frac{\text{absolute force of S}}{\text{S T}^2}$, and the numerators and denominators of the last terms are changed in the same given ratio, the accelerating forces remain in the same ratio as before, and the linear or angular errors $\propto$ as before,

i. e. as the diameter of the orbits, and the times of those errors $\propto$ P T's of the bodies.

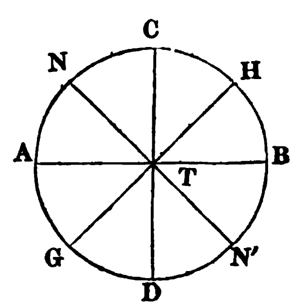
COR. 16. Hence if the forms and inclinations of the orbits remain, and the magnitude of the forces and the distances of the bodies be changed; to find the variation of the errors and the times of the errors. In Cor. 14, it was shown, how that when P T remained constant, the errors $\propto \frac{1}{P^2}$. Now let P T also $\propto$, then since the addititious force in a given position of P $\propto$ P T, and in a given position of P the addititious : ablatitious in a given ratio.

COR. If a body in an ellipse be acted upon by a force which varies in a ratio greater than the inverse square of the distance, it will in descending from the higher apse B to the lower apse A, be drawn nearer to the center. $\therefore$ as S is fixed, the excentricity is increased, and from A to B the excentricity will be increased also, because the force decreases the faster the distance[2] increases.



484. (COR. 10.) Let the plane of P's orbit be inclined to the plane of T's orbit remaining fixed. Then the addititious force being parallel to P T, is in the same plane with it, and $\therefore$ does not alter the inclination of the plane. But the ablatitious force acting from P to S may be resolved into two, one parallel, and one perpendicular to the plane of P's orbit. The force perpendicular to P's orbit $= 3\,A \times \sin.\,\theta \times \sin.\,Q \times \sin.\,T$ when $\theta =$ perpendicular distance of P from the quadratures, $Q =$ angular distance of the line of the nodes from the syzygy, $T =$ first inclination of the planes.

Hence when the line of the nodes is in the syzygy, $\theta = 0$, $\therefore \sin. = 0$ $\therefore$ no force acts perpendicular to the plane, and the inclination is not changed. When the line of the nodes is in the quadratures, $\theta = 90°$, $\therefore$ sin. is a maximum, $\therefore$ force perpendicular produces the greatest change in the inclination, and $\sin.\,\theta$ being positive from C to D, the force to change the inclination continually acts from C to D pulling the plane down from D to C. $\sin.\,\theta$ is negative, $\therefore$ force which before was posi-



tive pulling down to the plane of S's orbit (or to the plane of the paper) now is negative, and ∴ pulls up to the plane of the paper. But P's orbit is now below the plane of the paper, ∴ force still acts to change the inclination. Now since the force from C to D continually draws P towards the plane of S's orbit, P will arrive at that plane before it gets to D.

If the nodes be in the octants past the quadrature, that is between C and A. Then from N to D, sin. θ being positive, the inclination is diminished, and from D to N′ increased, ∴ inclination is diminished through 270°, and increased through 90°, ∴ in this, as in the former case, it is more diminished than increased. When the nodes are in the octants before the quadratures, i. e. in G H, inclination is decreased from H to C, diminished from C to N, (and at N the body having got to the highest point) increased from N to D, diminished from D′ to N′, and increased from 2 N′ to H, ∴ inclination is increased through 270°, and diminished through 90°, ∴ it is increased upon the whole. Now the inclination of P's orbit is a maximum when the force perpendicular to it is a minimum, i. e. when (by expression) the line of the nodes is in the syzygies. When is the quadratures, and the body is in the syzygies, the least it is increased when the apsides move from the syzygies to the quadratures; it is diminished and again increased as they return to the syzygies.

485. (Cor. 11.) While P moves from the quadrature in C, the nodes being in the quadrature it is drawn towards S, and ∴ comes to the plane of S's orbit at a point nearer S than N or D, i. e. cuts the plane before it arrives at the node. ∴ in this case the line of the nodes is regressive. In the syzygies the nodes rest, and in the points between the syzygies and quadratures, they are sometimes progressive and sometimes regressive, but on the whole regressive; ∴ they are either retrograde or stationary.

486. (Cor. 12.) All the errors mentioned in the preceding corollaries are greater in the syzygies than in any other points, because the disturbing force is greater at the conjunction and opposition.

487. (Cor. 13.) And since in deducing the preceding corollaries, no regard was had to the magnitude of S, the principles are true if S be so great that P and T revolve about it, and since S is increased, the disturbing force is increased; ∴ irregularities will be greater than they were before.

488. (Cor. 14.) $L\,M = \frac{M\,a^2\,r}{d^3} = N\,N\,M = \frac{3\,M\,a^2\,r}{d^3}\sin.\theta$, ∴ in a given position of P, if P T remain unaltered, the forces N M and L M

$\propto \frac{1}{d^3} \times$ absolute force $\propto \frac{1}{(\text{Per. T})^2}$ of T for (sect. 3 . $P^2 \propto \frac{d^3}{\text{absolute f.}}$) whether the absolute force vary or be constant. Let D = diameter of S, δ = density of S, and attractive force of S $\propto$ magnitude or quantity of matter $\propto D^3 \delta$,

$$\therefore \text{ forces L M and N M} \propto \frac{D^3 \delta}{d^3}.$$

$$\text{But } \frac{D}{d} = \text{apparent diameter of S},$$

$$\therefore \text{ forces} \propto (\text{apparent diameter})^3 \delta \text{ another expression.}$$

489. (COR. 15.) Let another body as P′ revolve round T′ in an orbit similar to the orbit of P round T, while T′ is carried round S′ in an orbit similar to that of T round S, and let the orbit of P′ be equally inclined to that of T′ with the orbit P to that of T. Let A, a, be the absolute forces of S, T, A′, a′, of S′, T′,

$$\text{accelerating force of P by S : that of P by T} :: \frac{A}{SP^2} : \frac{a}{PT^2},$$

and the orbits being similar

$$\text{accelerating force of P}' \text{ by S}' : \text{that of P}' \text{ by T}' :: \frac{A'}{S'P'^2} : \frac{a}{P'T'^2},$$

$\therefore$ if $A' : a' :: A : a$, and the orbits being similar,

$$SP : PT^2 :: S'P' : P'T',$$

$$\text{accelerating force of P}' \text{ by S}' : \text{that of P}' \text{ by T}'$$

$$:: \text{force on P by S} : \text{force on P}' \text{ by T}',$$

and the errors due to the disturbing forces in the case of P are as $\frac{A}{ST^3} \times r$, in the case of P′ and S′ are as $\frac{A'}{S'T'^3} \times R$,

$$\therefore \text{ linear errors in the first case : that in the second} :: r : R.$$

$$\text{Angular errors} \propto \frac{\text{sin. errors}}{R},$$

$$\text{angular errors in the first case : that in the second} :: 1 : 1.$$

$$\text{Now Cor. 2. Lem. X. } T^2 \propto \frac{\text{linear errors}}{TC}$$

$$\propto \frac{\text{angular errors}}{R} \times R,$$

$$\therefore T^2 \propto \text{angular errors},$$

$$\therefore \text{ angular errors} : 360 :: T^2 : P^2,$$

$$\therefore T^2 \propto P^2 \times \text{angular errors},$$

$$\therefore T \propto P \text{ for} = \text{angular errors}.$$

490. (Cor. 16.) Suppose the forces of S, P T, S T to vary in any manner, it is required to compare the angular errors that P describes in similar, and similarly situated orbits. Suppose the force of S and T to be constant, $\therefore$ addititious force $\propto$ P T, $\therefore$ if two bodies describe in similar orbits = evanescent arcs. Linear errors $\propto p^2 \times PT$.

$\therefore$ angular errors $\propto p^2$ (p = per. time of P round T, P = that of T round S). But by Cor. 14. if P T be given, the absolute force of A and S T $\propto$.

$$\text{Angular errors} \propto \frac{1}{P^2}.$$

$\therefore$ if P T, S T and the absolute force alternately vary,

$$\text{angular errors} \propto \frac{p^2}{P^2},$$

$$\left(\begin{matrix} P = \text{per. time of P round T} \\ p = \text{per. time of T round S} \end{matrix}\right\} \text{force} \propto \frac{M a^2 r}{d^3},$$

$$\text{angular errors} \propto \frac{\text{linear errors}}{\text{radius}}.$$

$$\therefore \text{lin. errors} \propto \text{force } T^2 \propto \frac{M a^2 r}{d^3} \times P^2 \text{ by last Cor.}$$

$$\therefore \text{angular errors} \propto \frac{\frac{r P^2}{d^2 \times r}}{M a^2} \propto \frac{P^2}{p^2}).$$

Now the errors d t $\times$ p = whole angular errors $\propto \frac{p}{P^2}$.

$\therefore$ error d t $\propto \frac{p}{P^2}$ thence the mean motion of the apsides $\propto$ mean motion of the nodes, for each $\propto \frac{p}{P^2}$, for each error is formed by forces varying as proof of the preceding corollaries, both the disturbing forces, and $\therefore$ the errors produced by them in a given time will $\propto$ P T. Let P describe an indefinite small angle about T (in a given position of P), then the linear errors generated in that time $\propto$ force T P time^2, but the time of describing = angles about T $\propto$ whole periodic time (p), $\therefore$ linear errors $\propto$ P T p^2, and as the same is true for every small portion, similar; the linear errors during a whole revolution $\propto$ P T p^2. Angular errors $\propto \frac{\text{linear er.}}{\text{rad.}} \therefore \propto p^2 \therefore$ when S T, P T, and the absolute force vary, the angular errors $\propto \frac{p^2}{P^2} \propto \frac{\text{absolute } p^2}{ST^3} \propto \frac{p^2}{ST^3}$ (when the absolute force is

given.) Now the error in any given time$\times$p varies the whole errors during a revolution $\propto \frac{p^2}{P^2}$. $\therefore$ the errors in any given time $\propto \frac{p}{P^2}$. Hence the mean motion of the apsides of P's orbit varies the mean motion of the nodes, and each will $\propto \frac{p}{P}$, the excentricities and inclination being small and remaining the same.

491. (COR. 17.) To compare the disturbing forces with the force of P to T.

$$\text{F of S on T} : \text{F of P on T} :: \frac{\text{absolute F}}{\text{S T}^2} : \frac{\text{a}}{\text{T P}^2}$$

$$\text{P}^2 \propto \frac{\text{absolute F}}{\text{axis major}^3} \qquad :: \frac{\text{A . S T}}{\text{S S}^3} : \frac{\text{a T P}}{\text{T P}^3}$$

$$:: \frac{\text{S T}}{\text{P}^2} : \frac{\text{TP}}{\text{p}^2} :: \frac{\text{d}}{\text{P}^2} : \frac{\text{r}}{\text{p}^2}$$

$$\text{mean add. F} : \text{F of S on T} :: \frac{\text{M a}^2\,\text{r}}{\text{d}^3} : \frac{\text{M a}^2}{\text{d}^2} :: \text{r} : \text{d}$$

$\therefore$ mean add. F : F P on T : : $\text{p}^2 : \text{P}^2$.

492. To compare the densities of different planets.

Let P and P′ be the periodic times of A and B, r and r′ their distances from the body round which they revolve.

$$\text{F of A to S} : \text{F of B to S} :: \frac{\text{r}}{\text{P}^2} : \frac{\text{r}'}{\text{P}'^2}$$

$$\frac{\text{quantity of matter in A}}{\text{distance}^2} : \frac{\text{do. in B}}{\text{distance}^2} :: \frac{\text{D}^3\text{ of A} \times \text{density}}{\text{distance}^2} : \frac{\text{D}^3\text{ of B} \times \text{density}}{\text{distance}^2}$$

$$:: \frac{\text{r}}{\text{P}^2} : \frac{\text{r}'}{\text{P}'^2}$$

$$\therefore \frac{\text{D}^3 \times \text{d}}{\text{r}^3} : \frac{\text{D}'^3 \times \text{d}'}{\text{r}'^3} :: \frac{1}{\text{P}^2} : \frac{1}{\text{P}'^2}$$

$$\therefore \text{d} : \text{d}' :: \frac{\text{r}^3}{\text{D}^3\,\text{P}^2} : \frac{\text{r}'^3}{\text{D}'^3\,\text{P}'^2}$$

$$:: \frac{1}{\text{S}^3\,\text{P}^2} : \frac{1}{\text{S}'^3\,\text{P}'^2}$$

where S and S′ represent the apparent diameters of the two planets.

493. In what part of the moon's orbit is her gravity towards the earth unaffected by the action of the sun.

$$\text{F} = \frac{\text{M a}^2}{\text{r}} + \frac{\text{M a}^2\,\text{r}}{\text{d}^3} - \frac{3\,\text{M a}^2\,\text{r}}{\text{d}^3} \cdot \frac{1 - \cos.^2\theta}{2} + \frac{3}{2}\,\frac{\text{M a}^2\text{r}}{\text{d}^3}\sin.^2\theta$$

and when it is acted upon only by the force of gravity $= \frac{\text{M a}^2}{\text{r}}$ for the other forces then have no effect.

$$\therefore \frac{M a^2 r}{d^3} - \frac{3 M a^2 r}{d^3} \cdot \frac{1 - \cos. 2\theta}{2} + \frac{3}{2} \frac{M a^2 r}{d^3} \sin. \theta = 0$$

$$1 - 3 . \frac{1 - \cos. 2\theta}{2} + \frac{2}{3} \sin. 2\theta = 0$$

$$1 - \frac{3}{2} + \frac{3}{2} \cos. 2\theta + \frac{3}{2} \sin. 2\theta = 0$$

$$1 - \frac{3}{2} + \frac{3}{2} \cdot \frac{1 - \sin.^2 C}{2} + \frac{3}{2} \sin. 2\theta = 0$$

Let $x = \sin. \theta$,

$$(\therefore 1 - \tfrac{3}{2} + \tfrac{3}{2} - \tfrac{3}{2} \sin.^2 \theta + \tfrac{3}{2} \times 2 \sin. \theta \times \cos. \theta = 0)$$

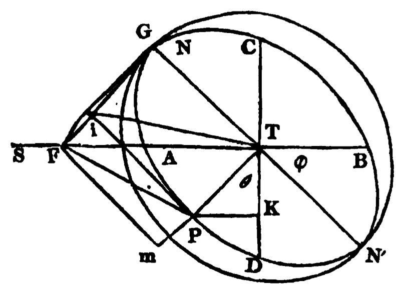
and

$$1 - \frac{3 x^2}{2} + 3 x \sqrt{1 - x^2} = 0.$$

An equation from which x may be found.

494. LEMMA. If a body moving towards a plane given in position, be acted upon by a force perpendicular to its motion tending towards that plane, the inclination of the orbit to the plane will be increased. Again, if the body be moving from the plane, and the force acts from the plane, the inclination is also increased. But if the body be moving towards the plane, and the force tends from the plane, or if the body be moving from the plane, and the force tends towards the plane, the inclination of the orbit to the plane is diminished.

495. To calculate that part of the ablatitious tangential force which is employed in drawing P from the plane of its orbit.



Let the dotted line upon the ecliptic N A P N′ be that part of P's orbit which lies above it. Let C D be the intersection of a plane drawn perpendicular to the ecliptic; P K perpendicular to this plane, and there-

fore parallel to the ecliptic. Take T F = 3 P K; join P F and it will represent the disturbing force of the sun. Draw P i a tangent to, and F i perpendicular to the plane of the orbit. Complete the rectangle i m, and P F may be resolved into P m, P i, of which P m is the effective force to alter the inclination. Draw the plane F G i perpendicular to N N′; then F G is perpendicular to N N′. Also F i G is a right angle. Assume P T tabular rad. Then

$$\left.\begin{array}{l} PT : TF :: R : 3g \\ TF : FG :: R : s \\ FG : Pm :: R : i \end{array}\right\} \begin{array}{l} \therefore PT : Pm :: R^3 : 3g.s.i \\ \therefore Pm = \dfrac{PT.3g.s.i}{R^3} \end{array}$$

$g = \sin.\theta = \sin.\angle$ dist. from quad.

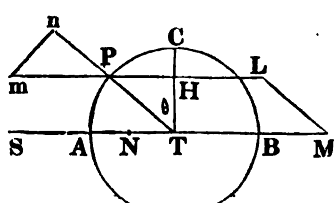
$s = \sin.\varphi = \sin.\angle$ dist. of nodes from syz.

$i = \sin. FTi = \sin. FGi =$ sin. inclination of orbit to ecliptic.

Hence the force to draw P from its orbit $= \dfrac{P.3gis}{R^3}$ when P is in the quadratures. Since g vanishes this force vanishes. When the nodes are in the syzygies s vanishes, and when in the quadratures this force is a maximum. Since s = rad. cotan. parte.

496. To calculate the quantity of the forces.

Let S T = d, P T = r, the mean distance from T = 1. The force



of T on P at the mean distance = f; the force of S on P at the mean distance = g.

Then the force S T $= \dfrac{g}{d^2}$, and the force S T : f. P T :: d : r,

$\therefore$ force P T $= \dfrac{gr}{d^3}$, hence the add. f $= \dfrac{gr}{d^3}$; ablat. f $= \dfrac{3gr}{d^3}\sin.\theta$, the mean add. force at distance 1 $= \dfrac{g}{d^3}$, the central ablat. $= \dfrac{3gr}{d^3}\sin.^2\theta$, the tangential ablat. f $= \dfrac{3gr}{2d^3}.\sin.2\theta$.

The whole disturbing force of S on P $= \frac{-g\,r}{2\,d^3} + \frac{3\,g\,r}{2\,d^3} \cdot \cos.\,2\,\theta$; the mean disturbing f $= \frac{-g\,r}{2\,d^3}$ (since cos. 2 θ vanishes) $= -\frac{m}{n}$ by supposition.

Hence we have the whole gravitation of P to T $= \frac{f}{r^2} - \frac{g\,r}{2\,d^3} + \frac{3\,g\,r}{2\,d^3} \times \cos.\,2\,\theta$, and the mean $= \frac{f}{r^2} - \frac{g\,r}{2\,d^2}$ (since cos. 2 θ vanishes).

PROBLEM.

497. Required the whole effect, and also the mean effect of the sun to diminish the lunar gravity; and show that if P and p be the periodic times of the earth and moon, f the earth's attraction at the mean distance of the moon, r the radius-vector of the moon's orbit; the addititious force will be nearly represented by the formula $\left\{\frac{p^2}{P^2} - \frac{p^4}{2\,P^4}\right\} f\,r$.

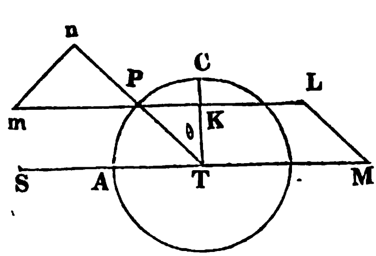
$P\,n = 3\,P\,T.\,\sin.^2\theta$, and $P\,T - 3\,P\,T.\sin.^2\theta = -\frac{P\,T}{2} + \frac{3}{2}\,P\,T \times \cos.\,2\,\theta$ = whole diminution of gravity of the moon, and the mean diminution $= -\frac{P\,T}{2} + \frac{-g\,r}{2\,d^3}$ by supposition.

Again,

$$P^2 \propto d^3$$

$$\therefore \frac{ab.\,f}{d^2} \propto \frac{d}{P^2}. \quad \text{Vid. seq.}$$

498. To find the central and ablatitious tangential forces.



Take $P\,m = 3\,P\,K = 3\,P\,T.\sin.\,\theta$ = ablatitious force.

Then $P\,n = P\,m.\sin.\,\theta = 3\,P\,T.\sin.^2\theta$ = central force

$m\,n = P\,m.\cos.\,\theta = 3\,P\,T.\sin.\,\theta.\cos.\,\theta$

$= \frac{3}{2}.\,P\,T\,\sin.\,2\,\theta$ = tangential ablatitious force.

To find what is the disturbing force of S on P.

The disturbing force $= P\,T - 3\,P\,T\,.\,\sin.^2\theta = \left(\frac{-1+3\cos.2\theta}{2}\right)\times$
$P\,T = -\frac{P\,T}{2} + \frac{3}{2}\,P\,T\,.\cos.\,2\,\theta.$

To find the mean disturbing force of S during a whole revolution.

Let P T at the mean distance = m, then $-\frac{P\,T}{2}+\frac{3}{2}\,.\,P\,T\cos.\,2\,\theta$
$= -\frac{P\,T}{2} = \frac{-m}{2}$ since cos. $2\,\theta$ is destroyed during a whole revolution.

499. To find the disturbing force in syzygy.

$$3\,A\,T - A\,T = 2\,A\,T = \text{disturbing force in syzygy;}$$

the force in quadrature is wholly effective and equal P T,

$\therefore$ force in quadrature : f in syzygy :: P T : 2 P T :: 1 : 2.

To find that point in P's orbit when the force of P to T is neither increased nor diminished by the force of S to T.

In this point $P\,n = P\,T$ or $3\,P\,T\sin.^2\theta = P\,T$,

$$\therefore \sin.\,\theta = \frac{1}{\sqrt{3}}$$

and

$$\theta = 35^\circ\ 16'.$$

To find when the central ablatitious force is a maximum.

$$P\,n = 3\,P\,T\,.\,\sin.^2\theta = \text{maximum},$$

$$\therefore d\,.\,(\sin.^2\theta) \text{ or } 2\sin.\,\theta\,.\cos.\,\theta - d\,\theta = 0,$$

$$\therefore \sin.\,\theta\,.\cos.\,\theta = 0,$$

or

$$\sin.\,\theta\,.\sqrt{1-\sin.^2\theta} = 0,$$

and

$$\sin.\,\theta = 1,$$

or the body is in opposition.

Then (Prop. LVIII, LIX,)

$$T^2 : t^2 :: S\,P : C\,P :: S + P : S$$

and

$$T^2 : t^2 :: A^3 : x^3$$

$$\therefore A^3 : x^3 :: S + P : S$$

and

$$A : x :: (S + P)^{\frac{1}{3}} : S^{\frac{1}{3}}.$$

500. PROB. Hence to correct for the axis major of the moon's orbit.

Let S be the earth, P the moon, and let per. t of a body moving in a secondary at the earth's surface be found, and also the periodic time of

the moon. Then we may find the axis major of the moon's orbit round the earth supposed at rest $= x$, by supposition. Then the corrected axis or axis major round the earth in motion : $x :: (S + P)^{\frac{1}{3}} : S^{\frac{1}{3}}$

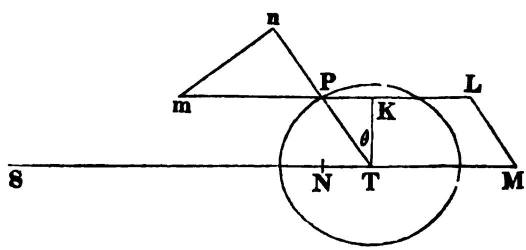
$$\therefore \text{ axis major round the earth in motion} = x \, . \, \frac{(S + P)^{\frac{1}{3}}}{S^{\frac{1}{3}}} = y.$$

Hence to compare the quantity of matter in the earth and moon,

$$y : x :: \sqrt[3]{S + P} : \sqrt[3]{S}$$

$$\therefore y^3 - x^3 : x^3 :: P : S.$$

501. To define the addititious and ablatitious forces. Let S T represent the attractive force of T to S. Take



$$SL : ST :: \frac{1}{SP^2} : \frac{1}{ST^2} :: ST^2 : SP^2$$

and S L will represent the attractive force of P to S. Resolve this into S M, and L M; then L M, that part of the force in the direction P T is called the addititious force, and $SM - ST = NM$ is the ablatitious force.

502. To compare these forces.

Since $SL : ST :: ST^2 : SP^2$, $\therefore SL = \frac{ST^3}{SP^2}$ = attractive force of P to S in the direction S P, and $SP : ST :: \frac{ST^3}{SP^2} : \frac{ST^4}{SP^3}$ = attractive force of P to S in the direction $TS = ST^4 (ST - PK)^{-} = ST + 3PK = SM$ nearly,

$$\therefore 3PK = TM = PL = \text{ablatitious force} = 3PT \, . \, \sin. \theta.$$

$$\text{Also } SP : PT :: \frac{ST^3}{SP^2} : \frac{ST^3}{SP^3}.$$

PT = attractive force of P to S in the direction $LM = PT$ nearly.

Hence the addititious force : ablatitious force $:: PT : 3PT \, . \, \sin. \theta : 1 : 3 \sin. \theta$. Q. e. d.

BOOK III.

1. Prop. I. All secondaries are found to describe areas round the primary proportional to the time, and these periodic times to be to each other in the sesquiplicate ratio of their radii. Therefore the center of force is in the primary, and the force $\propto \frac{1}{D^2}$.

2. Prop. II. In the same way, it may be proved, that the sun is the center of force to the primaries, and that the forces $\propto \frac{1}{\text{dist.}^2}$. Also the Aphelion points are *nearly* at rest, which would not be the case if the force varied in a greater or less ratio than the inverse square of the distance, by principles of the 9th Section, Book 1st.

3. Prop. III. The foregoing applies to the moon. The motion of the moon's apogee is very slow—about 3° 3′ in a revolution, whence the force will $\propto \frac{1}{\text{dist.}^{2\frac{4}{243}}}$. It was proved in the 9th Section, that if the ablatitious force of the sun were to the centripetal force of the earth :: 1 : 357.45, that the motion of the moon's apogee would be $\frac{1}{2}$ the real motion.

$\therefore$ the ablatitious force of the sun : centripetal force :: 2 : 357.45
:: 1 : 178 $\frac{29}{40}$.

This being very small may be neglected, the remainder $\propto \frac{1}{D^2}$.

4. Cor. The mean force of the earth on the moon : force of attraction :: 177 $\frac{29}{40}$: 178 $\frac{29}{40}$.

The centripetal force at the distance of the moon : centripetal force at the earth :: 1 : D^2.

5. Prop. IV. By the best observations, the distance of the moon from the earth equals about 60 semidiameters of the earth in syzygies. If the moon or any heavy body at the same distance were deprived of motion in the space of one minute, it would fall through a space = 16 $\frac{1}{12}$ feet. For the

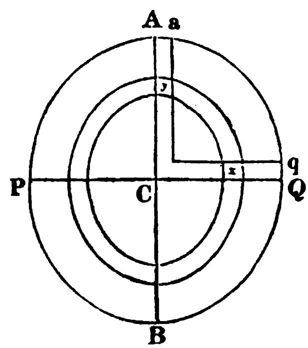
deflexion from the tangent in the same time = 16 $\frac{1}{12}$ feet. Therefore the space fallen through at the surface of the earth in 1″ = 16 $\frac{1}{12}$ feet.

$$\text{For } 60'' : t :: D : 1,$$

$$\therefore t = \frac{60''}{60} = 1'',$$

thence the moon is retained in its orbit by the force of the earth's gravity like heavy bodies on the earth's surface.

6. Prop. XIX. By the figure of the earth, the force of gravity at



the pole : force of gravity at the equator :: 289 : 288. Suppose A B Q q a spheroid revolving, the lesser diameter P Q, and A C Q q c a a canal filled with water. Then the weight of the arm Q q c C : ditto of A a c C : : 288 : 289. The centrifugal force at the equator, therefore I suppose $\frac{1}{289}$ of the weight.

Again, supposing the ratio of the diameters to be 100 : 101. By computation, the attraction to the earth at Q : attraction to a sphere whose radius = Q C : : 126 : 125. And the attraction to a sphere whose radius A C : attraction of a spheroid at A formed by the revolution of an ellipse about its major axis : : 126 : 125.

The attraction to the earth at A is a mean proportional between the attractions to the sphere whose radius = A C, and the oblong spheroid, since the attraction varies as the quantity of matter, and the quantity of matter in the oblate spheroid is a mean to the quantities of matter in the oblong spheroid and the circumscribing sphere.

Hence the attraction to the sphere whose radius = A C : attraction to the earth at A : : 126 : 125 $\frac{1}{2}$.

∴ attraction to the earth at the pole : attraction to the earth at the equator : : 501 : 500.

Now the weights in the canals ∝ whole weights ∝ magnitudes × gra-

vity, therefore the weight of the equatorial arm : weight of the polar

$$:: 500 \times 101 : 501 \times 100$$
$$:: \quad 505 \quad : 501.$$

Therefore the centrifugal force at the equator supports $\frac{4}{505}$ to make an equilibrium.

But the centrifugal force of the earth supports $\frac{1}{289}$,

$\therefore \frac{4}{505} : \frac{1}{289} :: \frac{1}{100} : \frac{1}{229} =$ the excess of the equatorial over the polar radius.

Hence the equatorial radius : polar $:: 1 + \frac{1}{229} : 1$

$$:: 230 : 229.$$

Again, since when the times of rotation and density are different the difference of the diameter $\propto \frac{V^2}{\text{dens.}}$, and that the time of the earth's rotation = 23h. 56′.

The time of Jupiter's rotation = 9h. 56′.

The ratio of the squares of the velocity are as 29 : 5, and the density of the earth : density of Jupiter :: 400 : 94.5.

d the difference of Jupiter's diameter is as $\frac{29}{5} \times \frac{400}{94.5} \times \frac{1}{229}$,

$$\therefore d : \text{Jupiter's least diameter} :: \frac{29}{5} \times \frac{400}{94.5} \times \frac{1}{229} :: 29 \times 80 : 94.5 \times 229$$
$$:: 2320 : 21640$$
$$:: 232 : 2164$$
$$:: 1 : 9\tfrac{1}{3}$$

The polar diameter : equatorial diameter :: $9\frac{1}{3}$: $10\frac{1}{3}$

ON THE TIDES.

7. THE PHENOMENA OF THE TIDES.

1. The interval between two succeeding high waters is 12 hours 25 minutes. The diminution varies nearly as the squares of the times from high water.

2. Twenty-four hours 50 minutes may be called the lunar day. The interval between two complete tides, the tide day. The first may be call-

ed the superior, the other inferior, and at the time of new moon, the morning and evening.

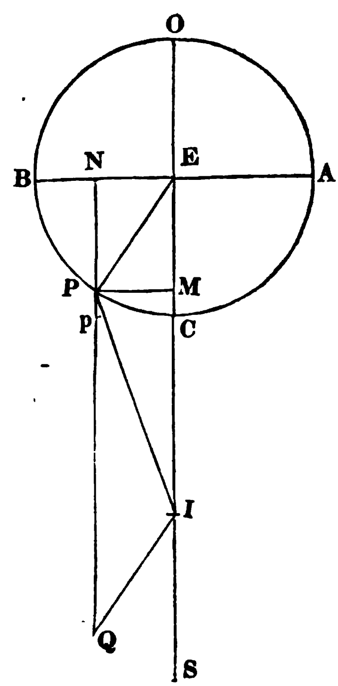
3. The high water is when the moon is in S.W. to us. The highest tide at Brest is a day and a half after full or change. The third full sea after the high water at the full moon is the highest; the third after quadrature is the lowest or neap tide.

4. Also the highest spring tide is when the moon is in perigee, the next spring tide is the lowest, since the moon is nearly in the apogee.

5. In winter the spring tides are greater than in summer, and from the same reasoning the neap tides are lower.

6. In north latitude, when the moon's declination is north, that tide in which the moon is above the horizon is greater than the other of the same day in which the moon is below the horizon. The contrary will take place if either the observer be in south latitude or the moon's declination south.

7. Prop. I. Suppose P to be any particle attracted towards a center E, and let the gravity of E to S be represented by E S. Draw B A perpendicular to E S, which will therefore represent the diameter of the plane of illumination. Draw Q P N perpendicular to B A, P M perpendicular to E C. Then take E I = 3 P N, and join P I, P I will represent the disturbing force of P. P I may be resolved into the two P E, P Q, of which P E is counterbalanced by an equal and opposite force, P Q acts in the direction N P.



Hence if the whole body be supposed to be fluid, the fluid in the canal N P will lose its equilibrium, and therefore cannot remain at rest. Now, the equilibrium may be restored by adding a small portion P p to the canal, or by supposing the water to subside round the circle B A, and to be collected towards O and C, so that the earth may put on the form of a prolate spheroid, whose axis is in the line O C, and poles in O and C, which may be

the case since the forces which are superadded $\propto$ N P, or the distance from B A, so that this mass may acquire such a protuberancy at O and C, that the force at O shall be to the force at B : : E A : E C; and by the above formula

$$\frac{x}{r} = \frac{5C}{4g} = \frac{EC - EA}{EA}.$$

8. Prop. II. Let W equal the terrestrial gravitation of C; G equal its gravitation to the sun; F the disturbing force of a particle acting at O and C; S and E the quantities of matter in the sun and earth.

$$\therefore F : W :: \frac{3S}{CS^2 \times CG} : \frac{C}{CE^3}$$

Since the gravitation to the sun $\propto \frac{1}{\text{dist.}^2}$

$$CS^2 : ES^2 :: ES : CG$$

$$\therefore CG \times CS^2 = ES^3.$$

$$\therefore F : W :: \frac{3S}{ES^3} : \frac{E}{CE^3}$$

and

$$E : S :: 1 : 338343$$

$$EC : ES :: 1 : 23668$$

$$\therefore \frac{3S}{ES^3} : \frac{E}{CE^3} :: 1 : 12773541 :: F : W.$$

$$\therefore 4W : 5F :: CE : EC - EA.$$

Attraction to the pole : attraction to the equator : : $1 - \frac{4d}{5} : 1 - \frac{3d}{5}$

Quantity of matter at the pole : do. at equator : : 1 : 1 — d.

Weight of the polar arm : weight of the equatorial arm : : $1 - \frac{4d}{5} : 1 - \frac{8d}{5}$

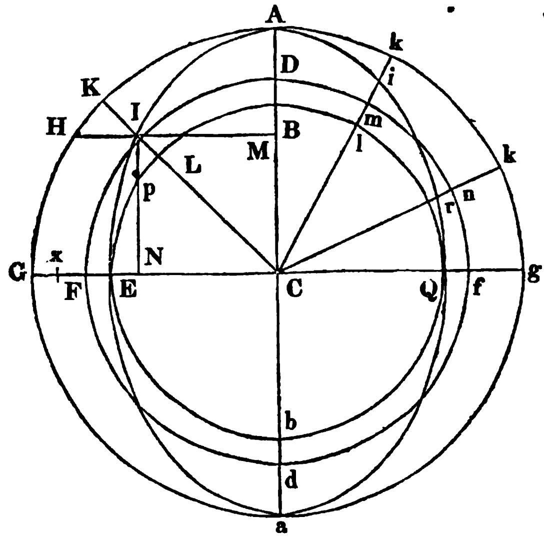
$$:: 1 + \frac{4d}{5} : 1$$

$\therefore$ Excess of the polar = attractive force : weight of the equator or mean weight W : : $\frac{4d}{5} : 1$

$$\therefore d = \frac{5F}{4W}.$$

9. Prop. III. Let A E a Q be the spheroid, B E b Q the inscribed

sphere, A G a g the circumscribed sphere, and D F d f the sphere equal (in capacity) to the spheroid.



Then since spheres and spheroids are equal to $\frac{2}{3}$ of their circumscribing cylinder, and that the spheroid = sphere D F d f.

$$CF^2 \times CD = CE^2 \times CA$$
$$CE^2 : CF^2 :: CD : CA,$$

and make

$$CE : CF :: CF : Cx$$
$$\therefore CE^2 : CF^2 :: CE : Cx$$
$$\therefore CD : CA :: CE : Cx$$
$$\therefore CD : CE :: CA : Cx$$

but

$$CD = CE \text{ nearly}$$
$$\therefore CA = Cx.$$

Also

$$Ex = 2\,EF \text{ nearly}$$
$$\therefore AD = 2\,EF.^*$$

* Let $CE = a$, $CF = a + x$,

$$\therefore Cx = \frac{a^2 + 2ax + x^2}{a} = \frac{a^2 + 2x}{a}$$
$$= a + 2x \text{ nearly}$$

$\therefore Ex = 2x$ nearly.

PROP. IV. By the triangles p I L, C I N,

$$A B : I L :: r^2 : (\cos.)^2 \angle T C A$$

$$\therefore I L = A B \times (\cos.)^2 \angle I C A = S \times (\cos.)^2 x$$

(if S = A B and x = angular distance from the sun's place.)

Again,

$$G E : K I :: r^2 : (\sin.)^2 \angle T C A$$

$$\therefore K I = S \times (\sin.)^2 \angle K.$$

COR. 1. The elevation of a spheroid above the level of the undisturbed ocean $= l\,i - l\,m = S \times (\cos.)^2 x - \frac{S}{3} = S \times \overline{(\cos.)^2 x - \frac{1}{3}}$.

The depression of the same $= S \times (\sin.)^2 x - S = S \times \overline{(\sin.)^2 x - \frac{2}{3}}$.

COR. 2. The spheroid cuts the sphere equal in capacity to itself in a point where $S \times (\cos.)^2 x = \frac{S}{3} = 0$, or $(\cos.)^2 x = \frac{1}{3}$.

$$\therefore \cos. x = .57734, \&c.$$
$$= \cos. 54^\circ. 44'.$$

10. PROP. V. The unequal gravitation of the earth to the moon is $(4000)^3$ times greater than towards the sun.

Let M equal the elevation above the inscribed sphere at the pole of the spheroid, γ equal the angular distance from the pole.

$\therefore$ the elevation above the equally capacious sphere $= M \times \overline{(\cos.)^2 \gamma - \frac{1}{3}}$

the depression ———————————— $= M \times \overline{(\sin.)^2 \gamma - \frac{2}{3}}$.

Hence the effect of the joint action of the sun and moon is equal to the sum or difference of their separate actions.

$\therefore$ the elevation at any place $= S \times (\cos.)^2 x + M \times (\cos.)^2 \gamma - \frac{1}{3}\,\overline{S+M}$

the depression ———— $= S \times (\sin.)^2 x + M \times (\sin.)^2 \gamma - \frac{2}{3}\,\overline{S+M}$.

1. Suppose the sun and moon in the same place in the heavens.

Then the elevation at the pole $= S + M - \frac{1}{3}\,\overline{S + M} = \frac{2}{3}\,\overline{S + M}$, and the depression at the equator $= S + M - \frac{2}{3}\,\overline{S + M} = \frac{1}{3}\,\overline{S + M}$,

$\therefore$ the elevation above the inscribed sphere $= S + M$.

2. Suppose the moon to be in the quadratures.

The elevation at $S = S - \frac{1}{3}\,\overline{S + M} = \frac{2}{3} S - \frac{1}{3} M$.

the depression at $M = S - \frac{2}{3}\,\overline{S + M} = \frac{1}{3} S - \frac{2}{3} M$,

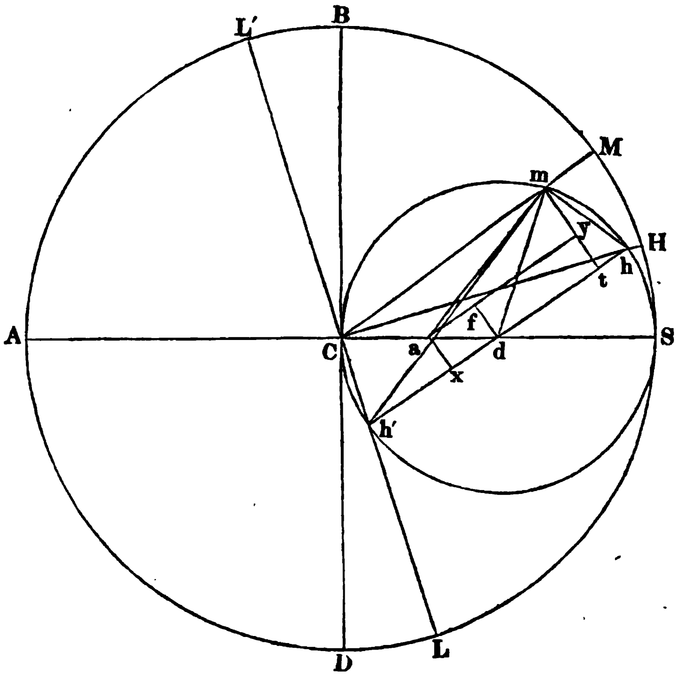
the elevation at S above the inscribed sphere $= S - M$,

the elevation at M (by the same reasoning) $= M - S$.

But (by observation) it is found that it is high water under the moon when it is in the quadratures, also that the depression at S is below the natural level of the ocean; hence M is more than twice S, and although

the high water is never directly under the sun or moon, when the moon is in the quadratures high water is always 6 hours after the high water at full or change.

Suppose the moon to be in neither of the former positions.



Then the place of high water is where the elevation = maximum,
or when $S \times \cos.^2 x + M \times \cos.^2 y$ = maximum,
and since

$$\cos.^2 x = \tfrac{1}{2} + \tfrac{1}{2} \cos. 2x,$$

and

$$\cos. y = \tfrac{1}{2} + \tfrac{1}{2} \cos. 2y,$$

elevation = maximum, when $S \times \cos. 2x + M \times \cos. 2y$ = maximum.

Therefore, let A B S D be a great circle of the earth passing through S and M, (those places on its surface which have the sun and moon in the zenith). Join C M, cutting the circle described on C S in (m). Make S d : d a : : force of the moon : force of the sun (which force is supposed

known). Join m a, m d, and let H be any point on the surface of the ocean. Join C H cutting the circle C m S in (h); draw the diameter h d h′, and draw m t, a x perpendicular to h h′, and a y parallel to it.

Then

$$M = S\,d,\ S = a\,d$$

and

$$\angle M C H = y,\ \angle S C H = x,$$
$$\therefore \angle m d h = 2 \angle M C H = 2 y$$

and

$$\angle a d x = \angle S d H = 2 x.$$
$$\therefore d t = M \times \cos. 2 y,\ d x = S \times \cos. 2 x,$$

$\therefore$ elevation = maximum when $t x = a y$ = maximum,
or when $a y = a m$, i. e. when $h h'$ is parallel to $a m$, hence

CONSTRUCTION.

Make

$$S d : d a :: M : S,$$

and join m a, draw h h′ parallel to a m, and from C draw C h H cutting the surface of the ocean in H, which is the point of high water.

Again, through h′ draw L C h′, meeting the circle in L, L′; these are the points of low water. For let

$$L C S = u,\ L C M = z.$$
$$\cos. \angle a d x = \cos. \angle S d h' = \cos. 2 \angle S C h' = \cos. 2 u = d x$$

and

$$\cos. 2 z = \cos. 2 L C M = d t.$$
$$\therefore S \times \cos. 2 u + M \times \cos. 2 z = \max.$$

Cor. If d f be drawn perpendicular to a m, a m represents the whole difference between high and low water, a f equals the point effected by the sun, m f that by the moon.

For

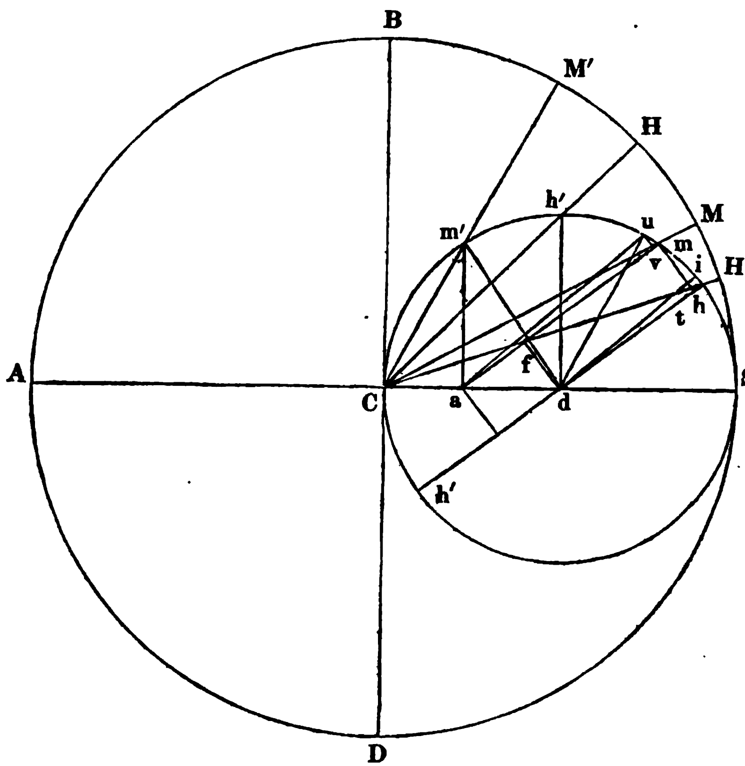
$$\sin.^2 u = \cos.^2 x,$$
$$\sin.^2 y = \cos.^2 x.$$

$$\therefore \text{elevation} + \text{depression} = S \times : \overline{\cos.^2 x - \tfrac{1}{3}} + M \times : \overline{\cos.^2 y - \tfrac{1}{3}} + S \times \overline{\cos.^2 x - \tfrac{2}{3}}$$
$$+ M \times : \cos.^2 y - \tfrac{2}{3} = S \times : \overline{2 \cos.^2 x - 1} + M \times : \overline{2 \cos.^2 y - 1}$$
$$= S \times \cos. 2 x + M \times \cos. 2 y$$

and

$$d t = M \times \cos. 2 y$$
$$d x = S \times \cos. 2 x.$$

12. **Conclusions deduced from the above (supposing that both the sun and moon are in the equator.)**



1. At new and full moon, high water will be at noon and midnight. For in this case C M, a m, C S, d h, C H, all coincide.

2. When the moon is in the quadrature at B, the place of high water is also at B under the moon, when the moon is on the meridian, for C M is perpendicular to C S, (m) coincides with C, (a m) with (a C), d h with d C.

3. While the moon passes from the syzygy to the quadrature the place of high water follows the moon's place, and is to the westward of it, overtakes the moon at the quadratures, and is again overtaken at the next syzygy. Hence in the first and third quadrants high water is after noon or midnight, but before the moon's southing, and in second and fourth *vice versâ*.

4. $\angle$ M C H = max. when S C H = 45°. S d h′ = 90°. and m′ a perpendicular to S C, and $\angle$ a m′ d = max., and a m′ d = m′ d h′ = 2 y′.

Hence in the octants, the motion of the high water = moon's easterly motion; in syzygy it is slower; in quadratures faster. Therefore the tide day in the octants = 24h. 50′ = the lunar day; in syzygy it is less = 24h. 35′; in quadratures = 25h. 25′.

For take any point (u) near (m), draw u a, u d, and d i parallel to a u and with the center (a) and radius a u, describe an arc (u v) which may be considered as a straight line perpendicular to a m; u m and i h are respectively equal to the motions of M and H, and by triangles u m v, d m f.

$$um : ih :: ma : mf.$$

Therefore the synodic motion of the moon's place : synodic motion of high water : : m a : m f.

Cor. 1. At new or full moon, m a coincides with S a, and m f with S d; at the quadratures, m a coincides with C a, and m f with C d; therefore the retardation of the tides at new or full moon : retardation at quadratures : : S a : C a : : M + S : M — S.

Cor. 2. In the octants, m a is perpendicular to S a, therefore m a, m f coincide. Therefore the synodic motion of high water equals the synodic motion of the moon.

Cor. 3. The variation of the tide during a lunation is represented by (m a); at S, m a = S a, at C = C a.

Therefore the spring tide : neap tide : : M + S : M — S.

Cor. 4. The sun contributes to the elevation, till the high water is in the octants, after which (a f) is — v e, therefore the sun diminishes the elevation.

Cor. 5. Let m u be a given arc of the moon's synodic motion, m v is the difference between the tides m a, u a corresponding to it.

Therefore by the triangles m u v, m d f.

$$mu : mv :: md : df.$$
$$\therefore mv \propto df;$$

and since

$$md : df :: r : \sin. dmf :: r : \sin. mdh :: r : \sin. 2\,MCH$$
$$mv \propto \sin. 2 \text{ arc } MH.$$

13. Prop. VI. In the triangle m d a, m d, d a and ∠ m d a are known when the proportion M : S is known and the moon's elongation.

Let the angle m d a = a,

and make

$$M + S : M - S :: \tan. a : \tan. b$$

then

$$y = \frac{a - b}{2}, \; x = \frac{a + b}{2}.$$

For

$$\begin{aligned} M + S : M - S &: : m\,d + d\,a : m\,d - d\,a \\ &: : \tan. \frac{m\,a\,d + a\,m\,d}{2} : \tan. \frac{m\,a\,d - a\,m\,d}{2} \\ &: : \tan. \frac{2\,x + 2\,y}{2} : \tan. \frac{2\,x - 2\,y}{2} \\ &: : \tan. \overline{x + y} : \tan. \overline{x - y} \\ &: : \tan. a : \tan. b, \end{aligned}$$

$$\therefore x + y : x - y : : a : b,$$

$$\therefore 2\,x = a + b, \; 2\,y = a - b,$$

$$\therefore x = \frac{a + b}{2}$$

and

$$y = \frac{a - b}{2}.$$

14. Prop. VII. To find the proportion between the accelerating forces of the moon and sun. 1st. By comparing the tide day at new and full moon with the tide day at quadratures.

$$35 : 85 : : M : S,$$

$$\therefore M : M : : \frac{35 + 85}{2} : \frac{85 - 85}{2} : : 5 : 2\tfrac{1}{12}.$$

Also, at the time of the greatest separation of high water from the moon in the triangle $m'\,d\,a$, $m\,d : d\,a : : r : \sin. 2\,y : : M : S$,

$$\therefore \frac{S}{M} = \sin. 2\,y,$$

at the octants y is found $= 12^\circ\, 30'$,

$$\therefore \frac{S}{M} = \sin. 25^\circ,$$

$$\therefore M : S : : 5 : 2\tfrac{1}{2} \text{ nearly.}$$

Hence taking this as the mean proportion at the mean distances of the moon and sun (if the earth $= 1$) the moon $= \frac{1}{70}$.

Cor. 1. If the disturbing forces were equal there would be no high or low water at quadratures, but there would be an elevation above the inscribed spheroid all round the circle, passing through the sun and moon $= \frac{1}{2}\, \overline{M + S}$.

Cor. The gravitation of the sun produces an elevation of 24 inches, the gravitation of the moon produces an elevation of 58 inches.

$\therefore$ the spring tide $=$ 82 inches, and the neap tide $=$ $33\frac{5}{4}$ inches.

15. Cor. 3. Though M : S : : 5 : 2, this ratio varies nearly from (6 : 2) to 4 : 2, for supposing the sun and moon's distance each $=$ 1000.

In January, the distance of the sun $=$ 983, perigee distance of the moon $=$ 945.

In July, the distance of the sun $=$ 1017, apogee distance of the moon $=$ 1055.

Disturbing force $\propto \frac{1}{D^3}$; hence

	S	M
apogee	1.901	4.258
mean	2	5
perigee	2.105	5.925.*

The general expression is $M = \frac{5}{2} S \times \frac{\Delta^3}{D^3} \times \frac{d^3}{\delta^3}$.

To find the general expression above.

Disturbing force of different bodies (See Newton, Sect. 11th, p. 66, Cor. 14.) $\propto \frac{1}{D^3}$,

$\therefore$ disturbing force S : disturbing force at mean distance : : $D^3 : \Delta^3$
disturbing force M : disturbing force at mean distance : : $d^3 : \delta^3$,

$$\therefore \frac{M}{S} : \frac{5}{2} : : \frac{d^3}{D^3} : \frac{\delta^3}{\Delta^3},$$

$$\therefore \frac{M}{S} = \frac{5}{2} \times \frac{\Delta^3}{D^3} \times \frac{d^3}{\delta^3},$$

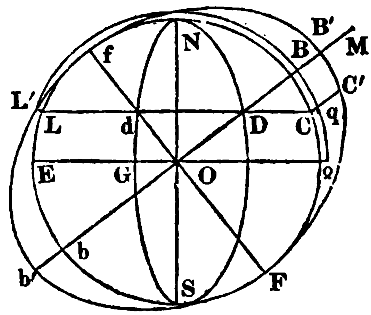
$$\therefore M = \frac{5}{2} \times S \times \frac{\Delta^3}{D^3} \times \frac{d^3}{\delta^3}$$

(or supposing that the absolute force of the sun and moon are the same).

16. Prop. VIII. Let N Q S E be the earth, N S its axis, E Q its equator, O its center; let the moon be in the direction O M having the declination B Q.

* The solar force may be neglected, but the variation of the moon's distance, and proportionally the variation of its action, produces an effect on the times, and a much greater on the heights of the tides.

Let D be any point on the surface of the earth, D C L its parallel of latitude, N D S its meridian; and let B′ F b′ f be the elliptical spheroid of the ocean, having its poles in O M, and its equator F O f.



As the point D is carried along its parallel of latitude, it will pass through all the states of the tide, having high water at C and L, and low water when it comes to (d) the intersection of its parallel of latitude with the equator of the watery spheroid.

Draw the meridian N d G cutting the terrestrial equator in G. Then the arc Q G (converted into lunar hours) will give the duration of the ebb of the superior tide, G E in the same way the flood of the inferior. N. B., the whole tide G Q C′, consisting of the ebb Q G, and the flood G Q is more than four times G O greater than the inferior tide.

Cor. If the spheroid touch the sphere in F and f, C C′ is the height at C, L L′ the height at L, hence if L′ q be a concentric circle C′ q will be the difference of superior and inferior tides.

CONCLUSIONS DRAWN FROM PROP. VIII.

1. If the moon has no declination, the duration of the inferior and superior tides is equal for one day over all the earth.

2. If the moon has declination, the duration of the superior will be longer or shorter than the duration of the inferior according as the moon's declination and the latitude of the place are of the same or different denominations.

3. When the moon's declination equals the colatitude or exceeds it,

there will only be a superior or inferior tide in the same day, (the parallel of latitude passing through f or between N and f.)

4. The sin. of arc G O = tan. of latitude × tan. declination.

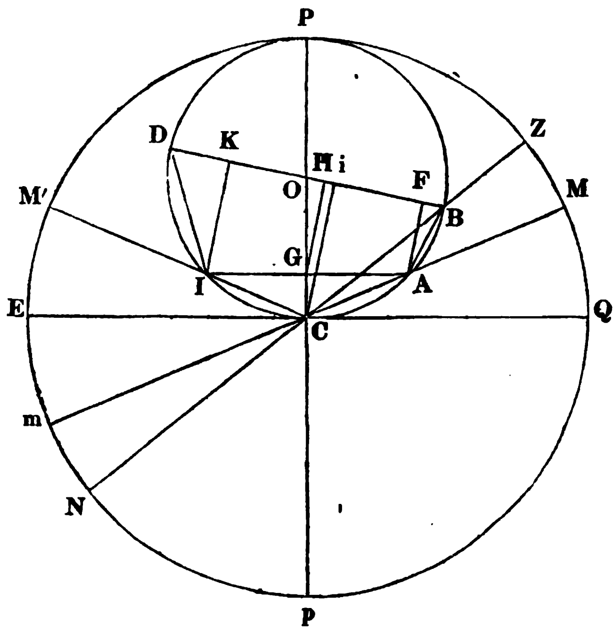
For

rad. : cot. d O G : : tan. d G : sin. G O,

∴ sin. G O = cot. d O G × tan. G d

= tan. declination × tan. latitude.

17. Prop. IX. With the center C and radius C Q (representing the



whole elevation of the lunar tide) describe a circle which may represent the terrestrial meridian of any place, whose poles are P, p, and equator E Q. Bisect P C in O, and round O describe a circle P B C D; let M be the place on the earth's surface which has the moon in its zenith, Z the place of the observer. Draw M C m, cutting the small circle in A, and Z C N cutting the small circle in B; draw the diameter B O D and A I parallel to E Q, draw A F, G H, I K perpendicular to B D, and join I D, A B, A D, and through I draw C M' cutting the meridian in M'. Then after $\frac{1}{4}$ a diurnal revolution the moon will come into the situation M', and the angle M' C N (= the nadir distance) = supplement the angle I C B = ∠ I D B.

Also the ∠ A D B = B C A = zenith distance of the moon.

Hence D F, D K $\propto$ cos.^{2} of the zenith and nadir distances to rad. D B.
$\propto$ elevation of the superior and inferior tides.

CONCLUSIONS FROM PROP. IX.

1. The greatest tides are when the moon is in the zenith or nadir of the observer. For in this case (when M approaches to Z) A and I move towards D, B, and F coincides with B; but in this case, the medium tide which is represented by D H (an arithmetic mean to D K, D F) is diminished.

If Z approach to M, D and I separate; and hence, the superior and inferior and the medium tides all increase.

2. If the moon be in the equator, the inferior and superior tides are equal, and equal M $\times$ (cos)^2 latitude. For since A and I coincide with C, and F and K with (i) D i $\doteq$ D B $\times$ (cos.)^2 B D C $=$ M $\times$ (cos.)^2 latitude.

3. If the observer be in the equator, the superior and inferior tides are equal every where, and $=$ M $\times$ (cos.)^2 of the declination of the moon. For B coincides with C, and F and K with G; P G $=$ P C $\times$ cos.^{2} of the moon's declination $=$ M $\times$ (cos.)^2 of the moon's declination.

4. The superior tides are greater or less than the inferior, according as the moon and place of the observer are on the same or different sides of the equator.

5. If the colatitude of the place equal the moon's declination or is less than it, there will be no superior or inferior tide, according as the latitude and the declination have the same or different denominations. For when P Z $=$ M Q, D coincides with I, and if it be less than M Q, D falls between I and C, so that Z will not pass through the equator of the watery spheroid.

6. At the pole there are no diurnal tides, but a rise and subsidence of the water twice in the month, owing to the moon's declining to both sides of the equator.

18. Prop. X. To find the value of the mean tide.

$$A\,G = \text{sin. } 2 \text{ declination (to rad. } = O\,C.)$$

and

$$O\,G = \text{cos. } 2 \text{ declination (to the same radius).}$$

$$\therefore O\,H = \text{cos. } 2 \text{ declination} \times \text{cos. } 2 \text{ lat.} \times \frac{M}{2},$$

$$\therefore D\,H = O\,D + O\,H$$

$$= M \times \frac{1 + \text{cos. } 2 \text{ lat.} \times \text{cos. } 2 \text{ declination}}{2}.$$

Now as the moon's declination never exceeds 30°, the cos. 2 declination is always + v e, and never greater than $\frac{1}{2}$; if the latitude be less than 45°, the cos. 2 lat. is + v e, after which it becomes — v e.

Hence

1. The mean tide is equally affected by north and south declination of the moon.

2. If the latitude = 45°, the mean tide $\frac{1}{2}$ M.

3. If the lat. be less than 45°, the mean tide decreases as the declination increases.

4. If the latitude be greater than 45°, the mean tide decreases as the declination diminishes.

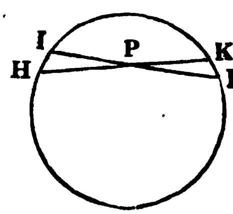
5. If the latitude = 0, the mean tide = $M \times \frac{1 + \text{cos. } 2 \text{ declination}}{2}$

BOOK I.

SECTION XII.

503. Prop. LXX. To find the attraction on a particle placed *within* a spherical surface, force $\propto^{s}. \frac{1}{\text{distance}^2}$.

Let P be a particle, and through P draw H P K, I P L making a very small angle, and let them revolve and generate conical surfaces I P H, L P K. Now since the angles at P are equal and the angles at H and L are also equal (for both are on the same segment of the circle), therefore the triangles H I P, P L K, are similar.



$$\therefore \text{H I} : \text{K L} :: \text{H P} : \text{P L}$$

Now since the surface of a cone $\propto$ (slant side)^2,

$\therefore$ surface intercepted by revolution of I P H : that of L P K $:: \text{P H}^2 : \text{P L}^2$
$:: \text{H I}^2 : \text{K L}^2$

and attractions of each particle in I P H : that of L P K $:: \frac{1}{\text{H P}^2} : \frac{1}{\text{P L}^2}$
$:: \frac{1}{\text{H I}^2} : \frac{1}{\text{K L}^2}$

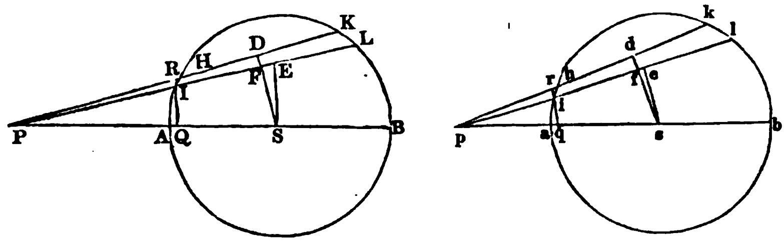
but the whole attraction of P $\propto$ the number of particles $\times$ attraction of each,

$\therefore$ the whole attraction on P from H I : from K L $:: \frac{\text{H I}^2}{\text{H I}^2} : \frac{\text{K L}^2}{\text{K L}^2}$
$:: 1 : 1;$

and the same may be proved of any other part of the spherical surface; $\therefore$ P is at rest.

504. Prop. LXXI. To find the attraction on a particle placed *without* a spherical surface, force $\propto^{s}. \frac{1}{\text{distance}^2}$.

Let A B, a b, be two equal spherical surfaces, and let P, p be two particles at any distances P S, p s from their centers; draw P H K,



P I L very near each other, and S F D, S E perpendicular upon them, and from (p) draw p h k, p i l, so that h k, i l may equal H K, I L respectively, and s f d, s e, i r perpendiculars upon them may equal S F D, S E, I R respectively; then ultimately $PE = PF = pe = pf$, and $DF = df$. Draw I Q, i q perpendicular upon P S, p s.

Now

$$\left.\begin{array}{l} PI : PF :: IR : DF \\ \text{and} \\ pf : pi :: df : ir \end{array}\right\} \therefore PI\ pf : pi \cdot PF :: IR : ir :: IH : ih$$

Again

$$\left.\begin{array}{l} PI : PS :: IQ : SF \\ \text{and} \\ ps : pi :: sf : iq \end{array}\right\} \therefore PI \cdot ps : pi \cdot PS :: IQ : iq$$

$$\therefore PI^2 \cdot pf \cdot ps : (pi)^2 \cdot PF \cdot PS :: IQ \cdot IH : iq \cdot ih$$

: : circumfer. of circle rad. $IQ \times IH$: circumfer. of circle rad $iq \times ih$

: : annulus described by revolution of I Q : that by revolution of i q.

Now

$$\text{attraction on 1st annulus : attraction on 2d} :: \frac{\text{1st annulus}}{\text{distance}^2} : \frac{\text{2d annulus}}{\text{distance}^2}$$

$$:: \frac{PI^2 \cdot pf \cdot ps}{PI^2} : \frac{(pi)^2 \cdot PF \cdot PS}{(pi)^2}$$

$$:: pf \cdot ps : PF \cdot PS.$$

And

$$\text{attraction on the annulus : attraction in the direction } PS :: PI : PQ$$

$$:: PS : PF$$

$$\therefore \text{attraction in direction } PS = pf \cdot ps \cdot \frac{PF}{PS}$$

$$\therefore \text{whole att}^n\text{. of P to S : whole att}^n\text{. of (p) to s} :: pf \cdot ps \cdot \frac{PF}{PS} : PF \cdot PS \cdot \frac{pf}{ps}$$

$$:: ps^2 : PS^2 :: \frac{1}{PS^2} : \frac{1}{ps^2}$$

and the same may be proved of all the annuli of which the surfaces are composed, and therefore the attraction of P $\propto \frac{1}{PS^2} \propto \frac{1}{\text{distance}^2}$ from the center.

Cor. The attraction of the particles within the surface on P equals the attraction of the particles without the surface.

For $KL : IH :: PL : PI :: LN : IQ$.

$\therefore$ annulus described by IH : annulus described by KL

$$:: IQ . IH : KL . LN :: PI^2 : PL^2$$

$\therefore$ attraction on the annulus IH : attraction on the annulus KL

$$:: \frac{PI^2}{PI^2} : \frac{PL^2}{PL^2} :: 1 : 1,$$

and so on for every other annulus, and one set of annuli equals the part within the surface, and the other set equals the part without.

506. Prop. LXXII. To find the attraction on a particle placed *without* a solid sphere, force $\propto \frac{1}{\text{distance}^2}$.

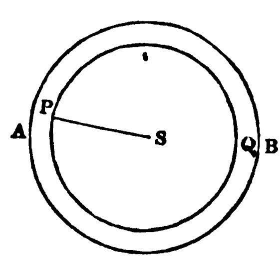
Let the sphere be supposed to be made up of spherical surfaces, and the attraction of these surfaces upon P will $\propto \frac{1}{\text{distance}^2}$, and therefore the whole attractions

$$\propto \frac{\text{number of surfaces}}{PS^2} \propto \frac{\text{content of sphere}}{PS^2} \propto \frac{\text{diameter}^3}{PS^2}$$

and if PS bear a given ratio to the diameter, then

$$\text{the whole attraction on P} \propto \frac{\text{diameter}^3}{\text{diameter}^2} \propto \text{diameter}.$$

507. Prop. LXXIII. To find the attraction on the particle placed *within*.



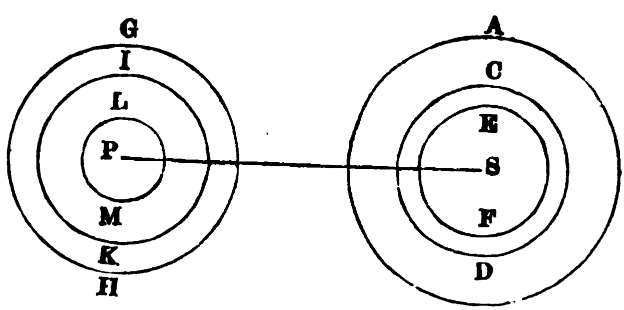
Let P be the particle; with rad. SP describe the interior sphere PQ; then by Prop. LXX. (considering the sphere to be made of spherical surfaces,) the attraction of all the particles contained between the circumferences of the two circles on P will be nothing, inasmuch as they are equal on each side of P, and the attraction of the other part by the last Prop. $\propto \frac{PS^3}{PS^2} \propto PS$.

1

508. PROP. LXXIV. If the attractions of the particles of a sphere $\propto \frac{1}{\text{distance}^2}$, and two similar spheres attract each other, then the spheres will attract with a force $\propto^g$ as $\frac{1}{\text{distance}^2}$ of their centers.

For the attraction of each particle $\propto \frac{1}{\text{distance}^2}$ from the center of the attracting sphere (A), and therefore with respect to the attracted particle the attracting sphere is the same as if all its particles were concentrated in its center. Hence the attraction of each particle in (A) upon the whole of (B) will $\propto \frac{1}{\text{distance}^2}$ of each particle in B from the center of P, and if all the particles in B were concentrated in the center, the attraction would be the same; and hence the attractions of A and B upon each other will be the same as if each of them were concentrated in its center, and therefore $\propto \frac{1}{\text{distance}^2}$.

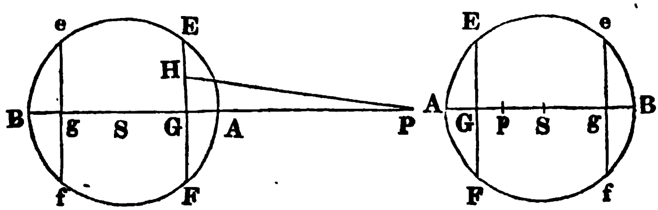
509. PROP. LXXVI. Let the spheres attract each other, and let them not be homogeneous, but let them be homogeneous at corresponding distances from the center, then they attract each other with forces $\propto^g . \frac{1}{\text{distance}^2}$.



Suppose any number of spheres C D and E F, I K and L M, &c. to be concentric with the spheres A B, G H, respectively; and let C D and I K, E F and L M be homogeneous respectively; then each of these spheres will attract each other with forces $\propto^g . \frac{1}{\text{distance}^2}$. Now suppose the original spheres to be made up by the addition and subtraction of similar and homogeneous spheres, each of these spheres attracting each

other with a force $\propto s . \frac{1}{\text{distance}^2}$; then the sum or differences will attract each other in the same ratio.

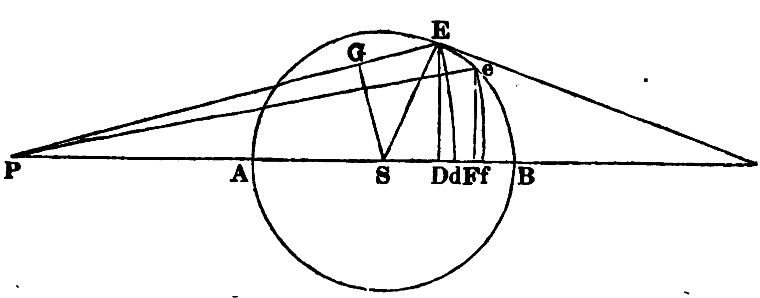
510. Prop. LXXVII. Let the force $\propto$ distance, to find the attraction of a sphere on a particle placed without or within it.



Let P be the particle, S the center, draw two planes E F, e f, equally distant from S; let H be a particle in the plane E F, then the attraction of H on P $\propto$ H P, and therefore the attraction in the direction S P $\propto$ P G, and the attraction of the sum of the particles in E F on P towards S $\propto$ circle E F . P G, and the attraction of the sum of the particles in (e f) on P towards S $\propto$ circle e f . P g, therefore the whole attraction of E F, e f, $\propto$ circle E F (P G + P g) $\propto$ circle E F . 2 P S, therefore the whole attraction of the sphere $\propto$ sphere $\times$ P S.

When P is within the sphere, the attraction of the circle E F on P towards S $\propto$ circle E F . P G, and the attraction of the circle (e f) towards S $\propto$ circle e f . P g, and the difference of these attractions on the whole attraction to S $\propto$ circle E F (P g — P G) $\propto$ circle E F . 2 P S. Therefore the whole attraction of the sphere on P $\propto$ sphere $\times$ P S.

511. Lemma XXIX. If any arc be described with the center S, rad.



S B, and with the center P, two circles be described very near each other

cutting, first, the circle in E, e, and P S in F, f; and E D, e d, be drawn perpendicular to P S, then ultimately,

$$D d : F f :: P E : P S.$$

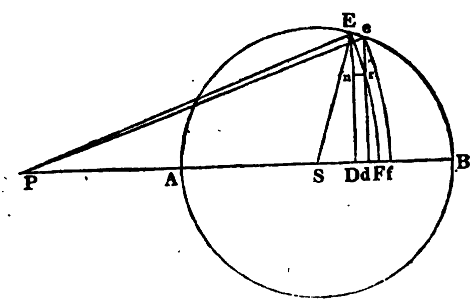
For

$$D d : E e :: D T : E T :: D E : E S$$

and

$$E e : F f :: E e : e r :: S E : S G$$
$$\therefore D d : F f :: \text{————} :: D E : S G :: P E : P S.$$

512. PROP. LXXIX. Let a solid be generated by the revolutions of an evanescent lamina E F f e round the axis P S, then the force with which the solid attracts P $\propto$ D E^2. F f $\times$ force of each particle.



Draw E D, e d perpendiculars upon P S; let e d intersect E F in r; draw r n perpendicular upon E D. Then E r : n r :: P E : E D, $\therefore$ E r . E D = n r . P S = D d . P E, $\therefore$ the annular surface generated by the revolution of E r $\propto$ E r . E D $\propto$ D d . P E, and (P E remaining the same) $\propto$ D d. But the attraction of this annular surface on P $\propto$ D d . P E, and the attraction in the direction P E : the attraction in the direction P S :: P E : P D,

$$\therefore \text{the attraction in the direction P S} \propto \frac{P D}{P E} . D d . P E \propto P D . D d$$

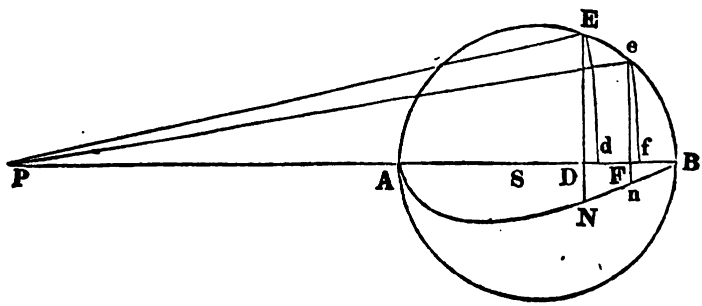
and the whole attraction on P of the surface described by E F $\propto$ sum of the P D . D d.

$$\text{Let } P E = r,\ D F = x,$$
$$\therefore P D = r - x,$$
$$\therefore P D . D d = r\,d x - x\,d x,$$

$$\therefore \text{sum of } P D . D d\ S = \int r\,d x - x\,d x = \frac{2 r x - x^2}{2} = \frac{D E^2}{2} \propto D E^2,$$

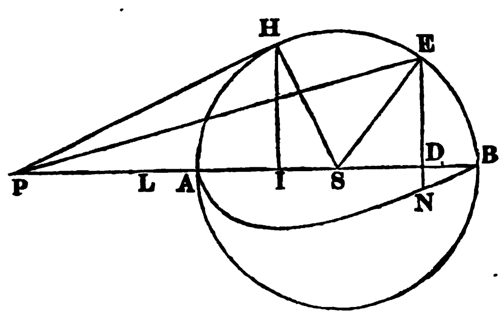
and therefore the attraction of lamina $\propto$ D E^2. F f $\times$ force of each particle.

513. Prop. LXXX. Take D N proportional to $\frac{DE^2 . PS}{PE} \times$ force of each particle at the distance P E, or if $\frac{1}{V}$ represent that force, let D N $\propto \frac{DE^2 . PS}{PE . V}$, then the area traced out by D N will be proportional to the whole attraction of the sphere.



For the attraction of lamina E F f e $\propto$ D E². F f $\times$ force of each particle $\propto$ (Lemma XXIII) $\frac{DE^2 . PS}{PE}$. D d $\times$ force of each particle, or $\propto \frac{DE^2 . PS}{PE . V}$ D d, $\therefore$ D N . D d $\propto$ attraction of lamina E F f e, and the sum of these areas or area A N B will represent the whole attraction of the sphere on P.

514. Prop. LXXXI. To find the area A N B.



Draw the tangent P H and H I perpendicular on P S, and bisect P I in L; then

$$PE^2 = PS^2 + SE^2 + 2PS.SD$$

But

$$SE^2 = SH^2 = PS.SI,$$

$$\begin{aligned}\therefore PE^2 &= PS^2 + PS.SI + 2PS.SD\\ &= PS\{PS + SI + 2SD\}\\ &= PS\{(PI + IS) + SI + 2SD\}\\ &= PS\{2LI + 2SI + 2SD\}\\ &= 2PS\{LI + SI + SD\} = 2PS.LD\end{aligned}$$

$$\begin{aligned}DE^2 &= SE^2 - SD^2 = SE^2 - (LD - LS)^2\\ &= SE^2 - LD^2 - LS^2 + 2LD.LS\\ &= 2LD.LS - LD^2 - (LS + SE)(LS - SE)\\ &= 2LD.LS - LD^2 - LB.LA,\end{aligned}$$

$$\therefore DN \propto \frac{DE^2.PS}{PE.V} \propto \frac{2LD.LS.PS}{\sqrt{2SD.PS}.V}$$

$$- \frac{LD^2.PS}{\sqrt{2LD.PS}.V} - \frac{LB.LA.PS}{\sqrt{2LD.PS}.V}$$

and hence if V be given, D N may be represented in terms of L D and known quantities.

515. Ex. 1. Let the force $\propto \frac{1}{\text{distance}}$; to find the area A N B.

Since $F \propto \frac{1}{\text{distance}} \propto \frac{1}{V}$, $\therefore V \propto PE$,

$$\therefore DN \propto \frac{2LS.LD.PS}{2LD.PS} - \frac{LD^2.PS}{2LD.PS} - \frac{AL.LB.PS}{2LD.PS}$$

$$\propto LS - \frac{LD}{2} - \frac{AL.LB}{2LD},$$

$$\therefore DN.Dd, \text{ or } d.\text{area} \propto LS.Dd - \frac{LD.Dd}{2} - \frac{AL.LB.Dd}{2LD},$$

$\therefore$ area A N D between the values of L A and L B

$$= LS.(LB - LA) - \frac{LB^2 - LA^2}{4} - \frac{AL.LB}{2}\,l\frac{LB}{LA}.$$

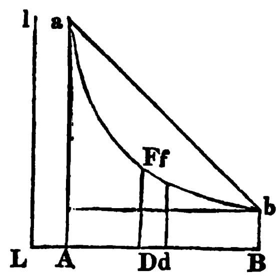
Now

$$\begin{aligned}LB^2 - LA^2 &= (LB + LA).(LB - LA)\\ &= (\overline{LS + AS} + \overline{LS - AS})\,AB = 2LS.AB,\end{aligned}$$

$$\therefore \text{area } AND = LS.AB - \frac{2LS.AB}{4} - \frac{AL.LB}{2}\,l\frac{LB}{LA}$$

$$= \frac{LS.AB}{2} - \frac{AL.LB}{2}\,l\frac{LB}{LA}.$$

516. To construct this area.



To the points L, A, B erect L l, A a, B b, perpendiculars, and let A a = L B, and B b = L A, through the points (a), (b), describe an hyperbola to which L l, L B are asymptotes. Then by property of the hyperbola, A L . A a = L D . D F,

$$\therefore DF = \frac{AL.Aa}{LD} = \frac{AL.LB}{LD},$$

$$\therefore DF.Dd = \frac{AL.LB.Dd}{LD},$$

$$\therefore \text{area } AaFD = \int DF.Dd = AL.LB \int LD,$$

$$\therefore \text{hyperbolic area } AafbB = AL.LB \int \frac{LB}{LA}.$$

$$\text{The area } AaBb = Bb.AB + \frac{AB.an}{2}$$

$$= \frac{Bb.AB}{2} + \frac{an+Bb}{2}.AB = \frac{Aa+Bb}{2}AB$$

$$= \frac{LB+LA}{2}.AB = LS.AB,$$

$$\therefore \text{area } afba = \text{area } AaBb - \text{area } AafbB$$

$$= LS.AB - AL.LB \int \frac{LB}{LA}.$$

517. Ex. 2. Let the force $\propto \frac{1}{\text{distance}^3}$; to find the area A N B.

$$\text{Let } V = \frac{PE^3}{2AS^2},$$

$$\therefore DN = \frac{2LS.LD.PS}{PE.V} - \frac{LD^2.PS}{PE.V} - \frac{AL.LB.PS}{PE.V}$$

but

$$V.PE = \frac{PE^4}{2AS^2} = \frac{4PS^2.LD^2}{2AS^2} = 2PS\frac{PS}{AS^2}.LD^2,$$

$$\therefore DN = \frac{SI.LS}{LD} - \frac{SI}{2} - \frac{AL.LB.SI}{2LD^2} = 2PS.\frac{1}{SI}.LD^2,$$

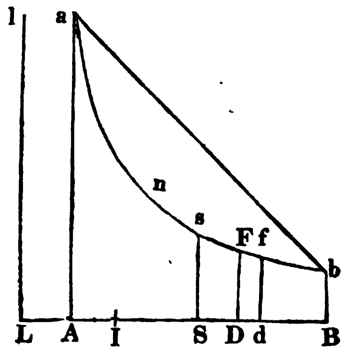
$$\therefore \int DN.x' = SI.LS \int LD - \frac{SI.LD}{2} + \frac{AL.LB.SI}{2LD},$$

$\therefore$ area between the values of L A and L B

$$= SI.LS \int \frac{LB}{LA} - \frac{SI.(LB-LA)}{2} - \left(\frac{LB.SI}{2} - \frac{AL.SI}{2}\right)$$

$$= SI.LS \int \frac{LB}{LA} - SI.AB.$$

To construct this area.



Take S I = S s, and describe a hyperbola passing through a, s, b, to which L l, L B are asymptotes; then as in the former case, the area A a n b B

$$= \mathrm{A\,L.\,S\,B.} \int \frac{\mathrm{L\,B}}{\mathrm{L\,A}} = \mathrm{L\,S.\,S\,s} \int \frac{\mathrm{L\,B}}{\mathrm{L\,A}} = \mathrm{S\,I.\,L\,S} \int \frac{\mathrm{L\,B}}{\mathrm{L\,A}}$$

$$\therefore \text{ the area A N B} = \mathrm{S\,I.\,L\,S} \int \frac{\mathrm{L\,B}}{\mathrm{L\,A}} - \mathrm{S\,I.\,A\,B}.$$

518. PROP. LXXXII. Let I be a particle within the sphere, and P the same particle without the sphere, and take

$$\mathrm{S\,P : S\,A :: S\,A : S\,I},$$

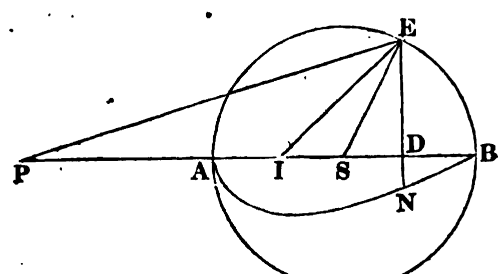
then will the attracting power of the sphere on I : attracting power of the sphere on P

$$:: \sqrt{\mathrm{S\,I}}.\ \sqrt{\text{force on I}} : \sqrt{\mathrm{S\,P}}.\ \sqrt{\text{force on P}}.$$

D N force on the point P : D′ N′ force on the point I

$$:: \frac{\mathrm{D\,E^2\,P\,S}}{\mathrm{P\,E.\,V}} : \frac{\mathrm{D\,E^2\,I\,S}}{\mathrm{I\,E.\,V'}}$$

$$:: \mathrm{P\,S.\,I\,E.\,V' : I\,S.\,P\,E.\,V}.$$



Let

$$\mathrm{V : V' :: P\,E^2 : I\,E^2},$$

then

$$DN : D'N' :: PS.IE.IE^n : IS.PE.P'E^n,$$

but

$$PS : SE :: SE : SI,$$

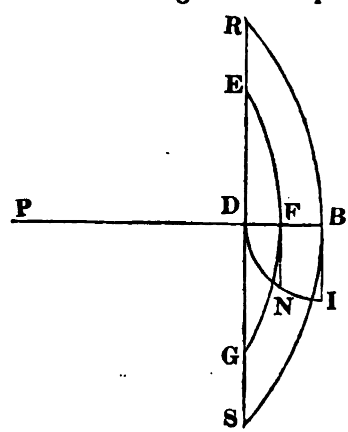
and the angle at S is common,

∴ triangles P S E, I S E are similar,

$$\therefore PE : IE :: PS : SE :: SE : SI,$$

$$\therefore DN : D'N' :: PS.SE.IE^n : PS.SI.PE^n,$$
$$:: SE.IE^n : SI.PE^n$$
$$:: \sqrt{SP}.IE^n : \sqrt{SI}.PE^n$$
$$:: \sqrt{SP} : SI^{\frac{n}{2}} \sqrt{SI}.PS^{\frac{n}{2}}.$$

519. Prop. LXXXIII. To find the attraction of a segment of a sphere upon a corpuscle placed within its centre.



Draw the circle F E G with the center P, let R B S be the segment of the sphere, and let the attraction of the spherical lamina E F G upon P be proportional to F N, then the area described by F N $\propto$ whole attraction of the segment to P.

Now the surface of the segment E F G $\propto$ P F D F, and the content of the lamina whose thickness is O $\propto$ PF D F O.

Let F $\propto \dfrac{1}{\text{distance}^n}$ and the attraction on P of the particle in that spherical lamina, $\propto$ (Prop. LXXIII.) $\dfrac{DE^2\,O}{PF^n}$

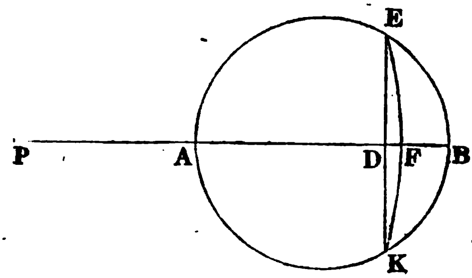
$$\propto \frac{(2\,PF\;FD - FD^2)\,O}{PF^n}$$

$$\propto \frac{2\,FD\,O}{PF^{n-1}} - \frac{FD^2\,O}{PF^n},$$

∴ if F N be taken proportional to $\dfrac{2\,FD}{PF^{n-1}} - \dfrac{FD^2}{PF^n}$, the area traced out by F N will be the whole attraction on P.

520. Prop. LXXXIV. To find the attraction when the body is placed in the axis of the segment, but not in the center of the sphere.

Describe a circle with the radius P E, and the segment cut off by the revolution of this circle E F K round P B, will have P in its center, and



the attraction on P of this part may be found by the preceding Proposition, and of the other part by PROP. LXXXI. and the sum of these attractions will be the whole attraction on P.

SECTION XIII.

521. PROP. LXXXV. If the attraction of a body on a particle placed in contact with it, be much greater than if the particle were removed at any the least distance from contact, the force of the attraction of the particles $\propto$ in a higher ratio than that of $\frac{1}{\text{distance}^2}$.

For if the force $\propto \frac{1}{\text{distance}^2}$, and the particle be placed at any distance from the sphere, then the attraction $\propto \frac{1}{\text{distance}^2}$ from the center of the sphere, and $\therefore$ is not sensibly increased by being placed in contact with the sphere, and it is still less increased when the force $\propto$ in a less ratio than that of $\frac{1}{\text{distance}^2}$, and it is indifferent whether the sphere be homogeneous or not; if it be homogeneous at equal distances, or whether the body be placed within or without the sphere, the attraction still varying in the same ratio, or whether any parts of this orbit remote from the point of contact be taken away, and be supplied by other parts, whether attractive or not, $\therefore$ so far as attraction is concerned, the attracting power of this sphere, and of any other body will not sensibly differ; $\therefore$ if the pheno-

mena stated in the Proposition be observed, the force must vary in a higher ratio than that of $\frac{1}{\text{distance}^2}$.

522. Prop. LXXXVI. If the attraction of the particles $\propto$ in a higher ratio than $\frac{1}{\text{distance}^3}$, or $\propto \frac{1}{\text{distance}^3}$, then the attraction of a body placed in contact with any body, is much greater than if they were separated even by an evanescent distance.

For if the force of each particle of the sphere $\propto$ in a higher ratio than that of $\frac{1}{\text{distance}^3}$, the attraction of the sphere on the particle is indefinitely increased by their being placed in contact, and the same is the case for any meniscus of a sphere; and by the addition and subtraction of attractive particles to a sphere, the body may assume any given figure, and ∴ the increase or decrease of the attraction of this body will not be sensibly different from the attraction of a sphere, if the body be placed in contact with it.

523. Prop. LXXXVII. Let two similar bodies, composed of particles equally attractive, be placed at proportional distances from two particles which are also proportional to the bodies themselves, then the accelerating attractions of corpuscles to the attracting bodies will be proportional to the whole bodies of which they are a part, and in which they are similarly situated.

For if the bodies be supposed to consist of particles which are proportional to the bodies themselves, then the attraction of each particle in one body : the attraction of each particle in the other body, : : the attraction of all the particles in the first body : the attraction of all the particles in the second body, which is the Proposition.

Cor. Let the attracting forces $\propto \frac{1}{\text{distance}^n}$, then the attraction of a particle in a body whose side is A : — B

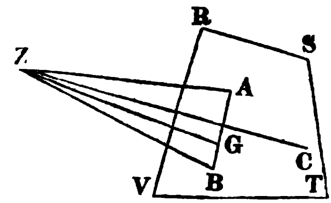
$$\therefore : \frac{A^3}{\text{distance}^n \text{ from } A} : \frac{B^3}{\text{distance}^n \text{ from } B}$$

$$:: \frac{A^3}{A^n} : \frac{B^3}{B^n}$$

$$:: \frac{1}{A^{n-3}} : \frac{1}{B^{n-3}},$$

if the distances $\propto$ as A and B.

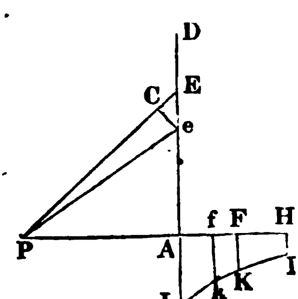
524. PROP. LXXXVIII. If the particles of any body attract with a force $\propto$ distance, then the whole body will be acted upon by a particle without it, in the same manner as if all the particles of which the body is composed, were concentrated in its center of gravity.



Let R S T V be the body, Z the particle without it, let A and B be any two particles of the body, G their center of gravity, then $A \cdot AG = B \cdot BG$, and then the forces of Z of these particles $\propto A \cdot AZ$, $B \cdot BZ$, and these forces may be resolved into $A \cdot AG + A \cdot GZ$, $B \cdot BG + B \cdot GZ$, and $A \cdot AG$ being $= B \cdot BG$ and acting in opposite directions, they will destroy each other, and $\therefore$ force of Z upon A and B will be proportional to $A \cdot ZG + B \cdot ZG$, or to $(A + B) ZG$, $\therefore$ particles A and B will be equally acted upon by Z, whether they be at A and B, or collected in their center of gravity. And if there be three bodies A, B, C, the same may be proved of the center of gravity of A and B (G) and C, and $\therefore$ of A, B, and C, and so on for all the particles of which the body is composed, or for the body itself.

525. PROP. LXXXIX. The same applies to any number of bodies acting upon a particle, the force of each body being the same as if it were collected in its center of gravity, and the force of the whole system of bodies being the same as if the several centers of gravity were collected in the common center of the whole.

526. PROP. XC. Let a body be placed in a perpendicular to the plane of a given circle drawn from its center; to find the attraction of the circular area upon the body.



With the center A, radius $=$ A D, let a circle be supposed to be described, to whose plane A P is perpendicular. From any point E in this circle draw P E, in P A or it produced take $PF = PE$, and draw F K perpendicular to P F, and let $FK \propto$ attracting force at E on P. Let I K L be the curve described by the point K, and let I K L meet A D in L, take $PH = PD$, and draw H I perpendicular

to P H meeting this curve in I, then the attraction on P of the circle $\propto$ A P the area A H I L.

For take E e an evanescent part of A D, and join P e, draw e C perpendicular upon P E, $\therefore$ E e : E C : : P E : A E, $\therefore$ E e . A E = E C $\times$ P E $\propto$ annulus described by A E, and the attraction of that annulus in the direction P A $\propto$ E C . P E . $\frac{PA}{PE} \times$ force of each particle at E $\propto$ E C $\times$ P A $\times$ force of each particle at E, but E C = F f, $\therefore$ F K . F f $\propto$ E C $\times$ the force of each particle at E, $\therefore$ attraction of the annulus in the direction P A $\propto$ P A . F f . F K, and $\therefore$ P A $\times$ sum of the areas F K . F f or P A the area A H I L is proportional to the attraction of the whole part described by the revolution of A E.

527. Cor. 1. Let the force of each particle $\propto \frac{1}{\text{distance}^2}$, at P F = x, let b = force at the distance a,

$$\therefore \text{F K the force at the distance } x = \frac{b a^2}{x^2},$$

$$\therefore FK . Ff = \frac{b a^2 dx}{x^2},$$

$$\therefore \text{attraction} = PA . FK . Ff = PA \int \frac{b a^2 dx}{x^2}$$

$$\propto PA - \frac{1}{x} \propto \quad A - \frac{1}{PF},$$

and between the values of P A and P H, the attraction

$$\propto PA \frac{1}{PA} - \frac{1}{PH} \propto 1 - \frac{PA}{PH}.$$

528. Cor. 2. Let the force $\propto \frac{1}{\text{distance}^n}$, then T K = $\frac{b a^n}{x^n}$,

$$\therefore \text{attraction} = PA \int \frac{b a^n}{x^n} dx \propto \frac{PA}{n-1} \times -\frac{1}{x^{n-1}} + \text{Cor.},$$

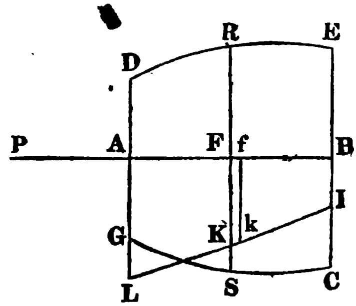
and between the values of P A and P H,

$$\text{attraction} = \frac{PA}{n-1} \left\{ \frac{1}{PA^{n-2}} - \frac{1}{PH^{n-1}} \right\}$$

$$\propto \frac{1}{PA^{n-1}} - \frac{PA}{PH^{n-1}}.$$

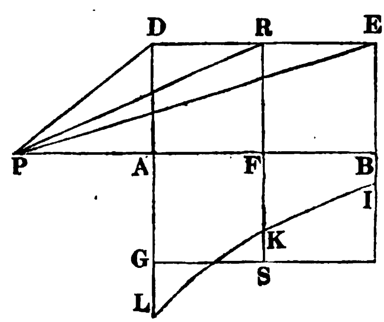
529. Cor. 3. Let the diameter of a circle become infinite, or P H $\propto \infty$, then the attraction $\propto \frac{1}{PA^{n-2}}$.

530. Prop. XCI. To find the attraction on a particle placed in the axis produced of a regular solid.



Let P be a body situated in the axis A B of the curve D E C G, by the revolution of which the solid is generated. Let any circle R F S perpendicular to the axis, cut the solid, and in the semidiameter F S of the solid, take F K proportional to the attraction of the circle on P, then F K . F f $\propto$ attraction of the solid whose base = circle R F S, and depth = F f, let I K L be the curve traced out by F K, $\therefore$ A L K F $\propto$ attraction of the solid.

COR. 1. Let the solid be a cylinder, the force varying as $\frac{1}{\text{distance}^2}$.



Then the attraction of the circle R F S, or F K which is proportional to that attraction $\propto 1 - \frac{PF}{PR}$.

Let P F = x, F R = b,

$$\therefore FK \propto 1 - \frac{x}{\sqrt{x^2 + b^2}},$$

$$\therefore FK \, . \, Ff \propto dx - \frac{x x'}{\sqrt{x^2 + b^2}},$$

$$\therefore \text{area} \propto -x \sqrt{x^2 + b^2}.$$

Now if $PA = x$, attraction $= 0$,

$\therefore$ Cor. $= PD - PA$,

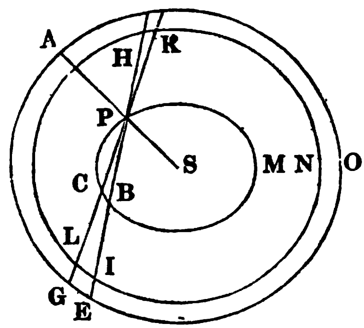
$\therefore$ whole attraction $= PB - PE + PD - PA$

$= AB - PE + PD.$

Let $AB = \infty = PE = PD$,

$\therefore$ atraction $= AB$.

531. Cor. 3. Let the body P be placed within a spheroid, let a spheroidical shell be included between the two similar spheroids D O G, K N I, and let the spheroid be described round S which will pass through P, and which is similar to the original spheroid, draw D P E, F P G, very near each other. Now $PD \doteq BE$, $PF = CG$, $PH = BI$, $PK = CL$.



$\therefore FK = LG$, and $DH = IE$,

and the parts of the spheroidical shell which are intercepted between these lines, are of equal thickness, as also the conical frustums intercepted by the revolution of these lines, and

$\therefore$ attraction on P by the part $DK : \ldots . GI$

$$:: \frac{\text{number of particles in } DK}{PD^2} : \frac{\ldots G^2}{PG^2}$$

$$:: \frac{PD^2}{PD^2} : \frac{PG^2}{PG^2} :: 1 : 1,$$

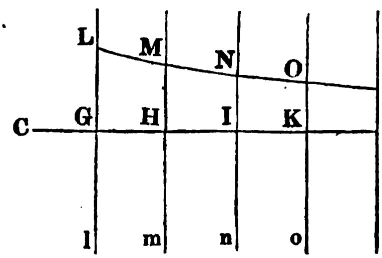
and the same may be proved of every other part of a spheroidical shell, and $\therefore$ body is not at all attracted by it; and the same may be proved of all the other spheroidical shells which are included between the spheroids, A O G, and C P M, and $\therefore$ P is not affected by the parts external to C P M, and $\therefore$ (Prop. LXXII.),

attraction on P : attraction on A :: P S : A S.

532. Prop. XCIII. To find the attraction of a body placed without an infinite solid, the force of each particle varying as $\frac{1}{\text{distance}^n}$, where n is greater than 3.

Let C be the body, and let G L, H M, K O, &c. be the attractions at the several infinite planes of which a solid is composed on the

body C; then the area G L O K equals the whole attraction of a solid on C.



Now if the force $\propto \frac{1}{\text{distance}^n}$.

Then

$$H M \propto \frac{1}{C H^{n-2}} \text{ (Cor. 3. Prop. XC.)}$$

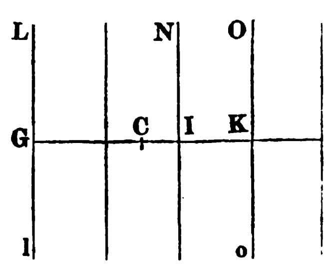
$$\therefore \int H M . d x \propto \int \frac{d x}{x^{n-2}} \propto -\frac{1}{x^{n-3}} + \text{Cor.}$$

$$\propto \frac{1}{C G^{n-3}} - \frac{1}{C H^{n-3}};$$

and if $H C = \infty$

then the area G L O K $\propto \frac{1}{G C^{n-3}}$.

Case 2. Let a body be placed within the solid.

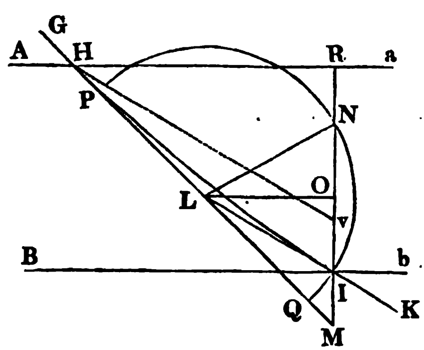


Let C be the place of the body, and take $C K = C G$; the part of the solid between G and K will have no effect on the body C, and therefore it is attracted to remain as if it were placed without it at the distance C K.

$$\therefore \text{attraction} \propto \frac{1}{C K^{n-3}} \propto \frac{1}{C G^{n-3}}.$$

SECTION XIV.

534. PROP. XCIV. Let a body move through a similar medium, terminated by parallel plane surfaces, and let the body, in its passage through this medium, be attracted by a force varying according to any law of its distance from the plane of incidence. Then will the sine of inclination be to the sine of refraction in a given ratio.



Let A a, B b be the planes which terminate the medium, and G H be the direction of the body's incidence, and I R that of its emergence.

Case 1. Let the force to the plane A a be constant, then the body will describe a parabola, the force acting parallel to I R, which will be a diameter of the parabola described. H M will be a tangent to the parabola, and if K I be produced I L will also be a tangent to the parabola at I. Let K I produced meet G M in L with the center L, and distance L I describe a circle cutting I R in N, and draw L O perpendicular to I R. Now by a property of the parabola M I = I v,

$$\therefore \text{M L} = \text{H L}, \ \therefore \text{M O} = \text{O R}, \text{ and } \therefore \text{M N} = \text{I R}.$$

The angle L M I = the angle of incidence, and the angle M I L = supplement of M I K = supplemental angle of emergence.

Now

$$\text{L} . \text{M I} = \text{M H}^2 = 4 \, \text{M L}^2$$

but

$$MN.MI=MI.IR=MQ.MP=\overline{ML+LQ}.\overline{ML-LQ}$$
$$=ML^2-LQ^2$$

$$\therefore MI=\frac{ML^2-LQ^2}{IR}$$

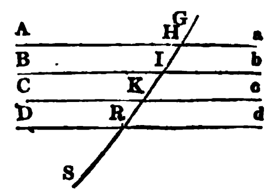
$$\therefore L:IR::4ML^2:ML^2-LQ^2$$

but L and I R are given

$$\therefore 4ML^2 \propto ML^2-LQ^2$$
$$\therefore ML^2 \propto LQ^2 \propto LI^2$$

$\therefore$ M L $\propto$ L I or sin. refraction : sin. inclination in a given ratio.

Case 2. Let the force vary according to any law of distance from A a.



Divide the medium by parallel planes A a, B b, C c, D d, &c. and let the planes be at evanescent distances from each other, and let the force in passing from A a to B b, from B b to C c, from C c to D d, &c. be uniform.

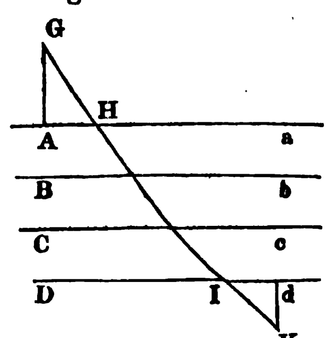
$\therefore$ sin. I at H : sin. R at H : : a : b

sin. R or I at I : sin. R at K : : c : d

sin. R or I at K : sin. R at R : : e : f, and so on.

$\therefore$ sin. I at H : sin. R at R : : a . c . e : b . d . f and in a constant proportion.

535. PROP. XCV. The velocity of a particle before incidence : velocity after emergence : : sin. emergence : sin. incidence.



Take A H = I d, and draw A G, d K perpendicular upon A a, D d, meeting the directions of incidence and emergence in G, K. Let the motion of the body be resolved into the two G A, A H, I d, d k, the ve-

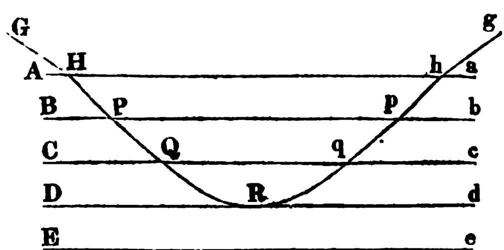
locity perpendicular to A a cannot alter the motion in the direction A a; therefore the body will describe G H, I K in the same time as the spaces A H, I d are described, that is, it will describe G H, I K in equal times before the incidence and after the emergence.

Velocity before incidence : velocity after emergence : : G H : I K

$$: : \frac{A\,H}{\text{sin. incidence}} : \frac{I\,d}{\text{sin. emergence}}$$

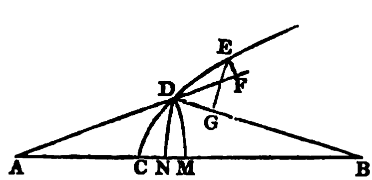
$$: : \text{sin. emergence} : \text{sin. incidence.}$$

536. Prop. XCVI. Let the velocity before incidence be greater than the velocity after emergence, then, by inclining the direction of the incident particle perpetually, the ray will be refracted back again in a similar curve, and the angle of reflection will equal the angle of incidence.



Let the medium be separated by parallel planes A a, B b, C c, D d, E e, &c. and since the velocity before incidence is greater than the velocity after emergence. ∴ sin. of emergence is greater than sin. of incidence. ∴ H P, P Q, Q R, &c. will continually make a less angle with H a, P b, Q c, R d, &c. till at last it coincides with it as at R; and after this it will be reflected back again and describe the curve R q p h g similar to R Q P H G, and the angle of emergence at h will equal the angle of incidence at H.

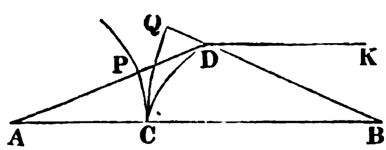
537. Prop. XCVII. Let sin. incidence : sin. refraction in a given ratio, and let the rays diverge from a given point; to find the surface of medium so that they may be refracted to another given point.



Let A be the focus of incident, B of refracted rays, and let C D E be the surface which it is required to determine. Take D E a small arc,

and draw E F, E G perpendiculars upon A D and D B; then D F, D G are the sines of incidence and refraction; or increment of A D : decrement of B D : : sin. incidence : sin. refraction. Take ∴ a point C in the axis through which the curve ought to pass, and let C M : C N : : sin. incidence : sin. refraction, and points where the circles described with radii A M, B N intersect each other will trace out the curve.

538. COR. 1. If A and B be either of them at an infinite distance or at any assigned situation, all the curves, which are the loci of D in different situations of A and B with respect to C, will be traced out by this process.



539. COR. 2. Describe circles with radii A C and C B, meeting A D, B D in P and Q; then P D : D Q : : sin. incidence : sin. refraction, since P D, D Q are the increments of B C and A C.

BOOK II.

SECTION I.

1. Prop. I. Suppose the resistance $\propto$ velocity, and supposing the whole time to be divided into equal portions, the motion lost will $\propto$ velocity, and $\propto$ space described. Therefore by composition, the whole decrement of the velocity $\propto$ space described.

Cor. Hence the whole velocity at the beginning of motion : that part which is lost : : the whole space which the velocity can describe : space already described.

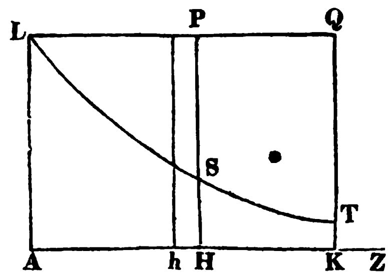
2. Prop. II. Suppose the resistance $\propto$ velocity.

Case 1. Suppose the whole time to be divided into equal portions, and at the beginning of each portion, the force of resistance to make a single impulse which will $\propto$ velocity, and the decrement of the velocity $\propto$ resistance in a given time, $\propto$ velocity. Therefore the velocities at the beginning of the respective portions of time will be in a continued progression. Now suppose the portions of time to be diminished *sine limite*, and then the number increased *ad infinitum*, then the force of resistance will act constantly, and the velocity at the beginning of equal successive portions of time will be in geometric progression.

Case 2. The spaces described will be as the decrements of the velocity $\propto$ velocity.

3. Cor. 1. Hence if the time be represented by any line and be divided into equal portions, and ordinates be drawn perpendicular to this line in geometric progression, the ordinates will represent the velocities, and the area of the curve which is the logarithmic curve, will be as the spaces described.

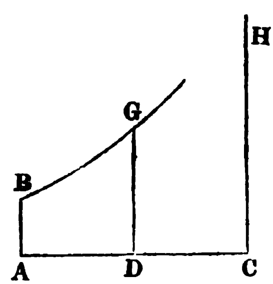
Suppose L S T to be the logarithmic curve to the asymptote A Z. A L, the velocity of the body at the beginning of the motion.



The space described in the time A H with the first velocity continued uniform : space described in the resisting medium, in the same time : : A H P L : area A L S H : : rect. A L × P L : rect. A L × P S*
: : P L : P S (if A L = subtan. of the curve).

Also since H S, K T representing the velocities in the times A H, A K; P S, Q T are the velocities lost, and therefore ∝ spaces described.

4. COR. 1. Suppose the resistance as well as the velocity at the beginning of the motion to be represented by the line C A, and after any time by the line C D. The area A B G D will be as the time, and A D as the space described.

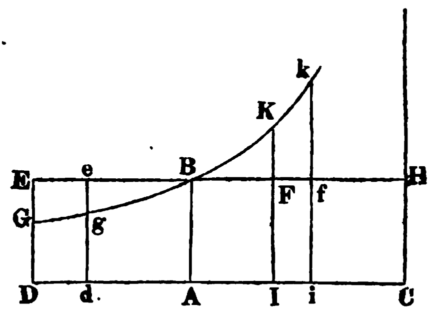


For if A B G D increase in arithmetical progression the areas being the hyperbolic logarithms of the abscissas, the abscissa will decrease in geometrical progression, and therefore A D will increase in the same proportion.

5. PROP. III. Let the force of gravity be represented by the rectangle

* Let the subtangent = M. Then the whole area of the curve = M × A L.
∴ the area A L S H = M × A L − M × H S = M × P S = A L × P S.

B A C H, and the force of resistance at the beginning of the motion by the rectangle B A D E on the other side of A B.

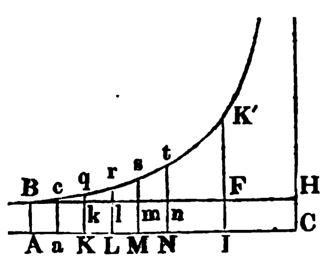


Describe the hyperbola G B K between the asymptotes A C and C H cutting the perpendiculars D E, d e, in G and g.

Then if the body ascend in the time represented by the area D G g d, the body will describe a space proportional to the area E G g e, and the whole space through which it can ascend will be proportional to the area E G B.

If the body descend in the time A B K I, the area described is B F K.

For suppose the whole area of the parallelogram B A C H to be di-



vided into portions, which shall be as the increments of the velocity in equal times, therefore A k, A l, A m, A n, &c. will $\propto$ velocity, and therefore $\propto$ resistances at the beginning of the respective times.

Let A C : A K : : force of gravity : resistance at the beginning of the second portion of time, then the parallelograms B A C H, k K C H, &c. will represent the absolute forces on the body, and will decrease in geometrical progression. Hence if the lines K k, L l, &c. be produced to meet

the curve in q, r, &c. these hyperbolic areas being all equal will represent the times, and also the force of gravity which is constant. But the area B A K q : area B q k : : K q : $\frac{1}{2}$ k q : : A C : $\frac{1}{2}$ A K : : force of gravity : resistance in the middle of the first portion of time.

In the same way, the areas q K L r, r L M s, &c. are to the areas q k l r, r l m s, &c. as the force of gravity to the force of resistance in the middle of the second, third, &c. portions of time. And since the first term is constant and proportional to the third, the second is proportional to the fourth, similarly as to the velocities, and therefore to the spaces described.

∴ by composition B k q, B r l, B s m, &c. will be as the whole spaces described, Q. e. d.

The same may be proved of the ascent of the body in the same way.

6. COR. 1. The greatest velocity which the body can acquire : the velocity acquired in any given time : : force of gravity : force of resistance at the end of the given time.

7. COR. 2. The times are logarithms of the velocities.

8. COR. 4. The space described by the body is the difference of the space representing the time, and the area representing the velocity, which at the beginning of the motion are mutually equal to each other.

Suppose the resistance to $\propto$ velocity.

$\therefore c^2 : v^2 : : r : \frac{r v^2}{c^2}$ = retarding force corresponding with the velocity (v)

$$\therefore v\, d v = -g \times \frac{r v^2}{c^2} \times d x,$$

$$\therefore d x = \frac{c^2}{g} \times \frac{dv}{v}$$

$$\therefore x = -b \times l\, v + C,$$

$$\therefore x = b \times l\, \frac{c}{v}$$

$$t = \frac{d x}{v} = -\frac{b dv}{v^2},$$

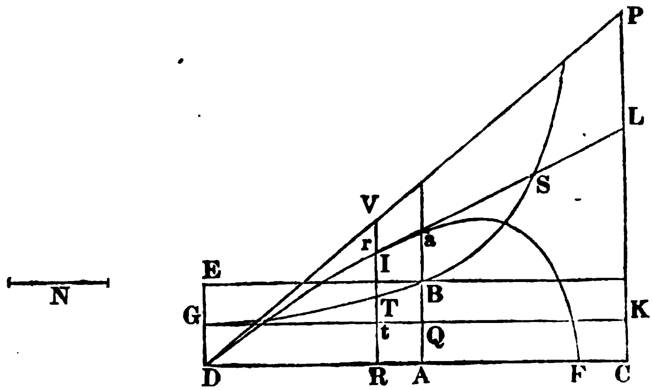
$$\therefore t = -b \times -\frac{1}{v} + \text{Cor.}$$

$$= b \times \overline{\frac{1}{v} - \frac{1}{c}} \propto \overline{\frac{1}{v} - \frac{1}{c}},$$

∴ the times being in geometrical progression, the velocities C, d, E, &c. will be in the same inverse geometrical progression.

Also the spaces will be in arithmetical progression.

9. Prop. IV. Let D P be the direction of the projectile, and let it represent the initial velocity; draw C P perpendicular to C D, and



let D A : A C : : resistance : gravity. Also D P : C P : : resistance : gravity, ∴ D A×D P : C P×C A : : R : G. Between D C, C P describe a hyperbola cutting D G and A B perpendicular to D C in G and B, from R draw R V perpendicular cutting D P in V and the hyperbola in T, complete the parallelogram G K C D and make N : Q B : : C D : C P.

Take

$$V\,r = \frac{G\,T\,t}{N} \text{ or } R\,r = \frac{G\,T\,E\,I}{N},$$

for since

$$N : Q\,B : : C\,D : C\,P : : D\,R : R\,V,$$

$$R\,V = \frac{D\,R \times Q\,B}{N},$$

and

$$\frac{G\,T\,E\,I}{N} = \frac{D\,R \times Q\,B - G\,T\,t}{N} = R\,r$$

in the time represented by D R T G the body will be at (r), and the greatest altitude = a, and the velocity ∝ r L.

For the motion may be resolved into two, ascending and lateral. The lateral motion is represented by D R, and the motion in ascent by R r, which

$$\propto D\,R \times Q\,B - G\,T\,t,$$

or

$$\frac{D\,R \times A\,B - D\,G\,.\,R\,T}{N},$$

the curve in q, r, &c. these l-
sent the times, and al-
area B A K q : are ... D K × A Q,
gravity : resistanc ... D K × A B — D K × A Q ... A B — A Q, or Q B

In the same ... D K : B T :: ... A B
q k l r, r l m s ... :: C D : C P,
dle of the s ... :: lateral motion . ascending motion at the beginning,
constant r ... :: (r) will be the place of the body required.
fourth,

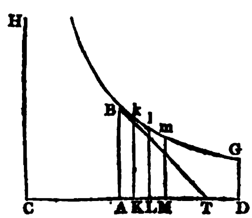
∴ '

des

SECTION II.

10. PROP. V. Suppose the resistance to vary as the velocity ^{2}.

Then as before, the decrement of velocity $\propto$ resistance $\propto$ velocity ^{2}.



Let the whole time A D be divided into a great number of equal portions, and draw the ordinates A B, K k, L l, M m, &c. to the hyperbola described between the two rectangular asymptotes, C H, C D; then by the property of the hyperbola,

$$A B : K k :: C K : C A,$$

$$\therefore A B - K k : K k :: A K : C A$$

$$:: A B \times A K : A B \times C A.$$

$$\therefore A B - K k \propto A B \times K k.$$

In the same way

$$K k - L l \propto K k^2, \&c.$$

or

$$A B^2, K k^2, L l^2, \&c.$$

are proportional to their differences.

∴ velocities will decrease in the same proportion. Also the spaces described are represented by the areas described by the ordinates; hence in

the time A M the space described may be represented by the whole area A M m B.

Now suppose the lines C A, C K, &c. and similarly A K, K L, &c. in geometrical progression, then the ordinates will decrease in the inverse geometrical progression, and the spaces will be *all* equal to each other.

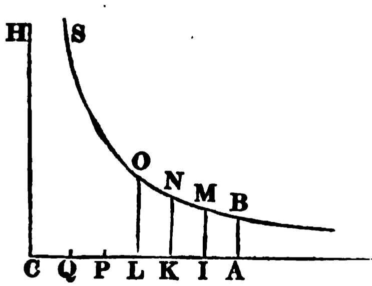
Q. e. d.

11. Cor. 1. The space described in the resisting medium : the space described with the first velocity continued uniform for the time A D : : the hyperbolic area A D G B : rectangle A B × A D.

12. Cor. 3. The first resistance equals the centripetal force which would generate the first velocity in the time A C, for if the tangent B T be drawn to the hyperbola at B, since the hyperbola is rectangular A T = A C, and with the first resistance continued uniform for the time A C the whole velocity A B would be destroyed, which is the time in which the same velocity would be generated by a force equal the first resistance. For the first decrement is A B — K k, and in equal times there would be equal decrements of velocity.

13. Cor. 4. The first resistance : force of gravity : : velocity generated by the force equal the first resistance in the time A C : velocity generated by the force of gravity in the same time.

14. Cor. 5. *Vice versâ*, if this ratio is given, every thing else may be found.



15. Prop. VIII. Let C A represent the force of gravity, A K the resistance, ∴ C K represents the absolute force at any time (if the body descend); A P, a mean proportional to A C and A K, represents the velocity; K L, P Q are contemporaneous increments of the resistance and the velocity.

Then since

$$A P^2 \propto A K, K L \propto 2 A P \times P Q \propto A P \times K C,$$

the increment of velocity $\propto$ force when the time is given,

$\therefore$ K L $\times$ K N $\propto$ A P $\times$ K C $\times$ K N,

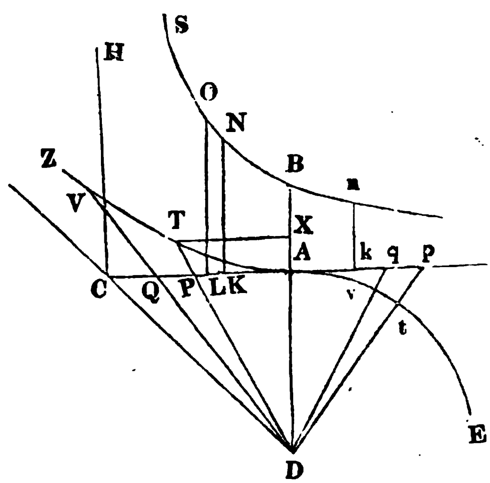
$\therefore$ ultimately K L O N (equal the increment of the hyperbolic area) $\propto$ A P $\propto$ velocity, $\propto$ space described, and the whole hyperbolic area = the sum of all the K L O Ns which are proportional to the velocity, and $\therefore$ space described. $\therefore$ If the whole hyperbolic area be divided into equal portions the absolute force C A, C I, C K, &c. are in geometrical progression. Q. e. d.

16. COR. 1. Hence if the space described be represented by a hyperbolic area, the force of gravity, velocity, and resistance, may be represented by lines which are in continued proportion.

17. COR. 2. The greatest velocity = A C.

18. COR. 3. If the resistance is known for a given velocity, the greatest velocity : given velocity : : $\sqrt{\text{force of gravity}}$: $\sqrt{\text{given resistance}}$.

19. PROP. IX. Let A C represent the greatest velocity, and A D be per-



pendicular and equal to it. With the center D and radius A D describe the quadrant A t E and the hyperbola A V Z. Draw the radii D P, D p. Then

Case 1. If the body ascend; draw D v q near to D p, $\therefore$ since the sector and the triangle are small,

$$\text{D v t} : \text{D p q} : : \text{D t}^2 : \text{D p}^2$$

$$\therefore \text{D v t} \propto \frac{\text{D q p}}{\text{D p}^2},$$

$$\propto \frac{\frac{1}{2} AD \times pq}{AD^2 + AD \times AK} \propto \frac{pq}{CK}$$

$\propto$ increment of the time.

$\therefore$ by composition, the whole sector $\propto$ whole time till the whole $V = 0$.

Case 2. If the body descend; as before

$$DVT : DPQ :: DT^2 : DP^2$$
$$:: DX^2 : DA^2 :: TX^2 : AP^2$$
$$:: DX^2 - TX^2 : DA^2 - AP^2$$
$$:: AD^2 : AD^2 - AD \times AK$$
$$:: AD : CK.$$

By the property of the hyperbola,

$$TX^2 = DX^2 - DA^2$$
$$\therefore DA^2 = DX^2 - TX^2$$
$$\therefore DVT \propto \frac{DPQ}{AD \times CK} \propto \frac{PQ}{Ck}$$

$\propto$ increment of the time.

$\therefore$ by composition, the whole time of descent till the body acquire its greatest $V =$ the whole hyperbolic sector DAT.

20. COR. 1. If $AB = \frac{1}{2} AC$.

The space which the descending body describes in any time : space which it would describe in a non-resisting medium to acquire the greatest velocity :: area $ABNK : \triangle ATD$, which represents the time. For since $AC : AP :: AP : AK$

$$KL : \tfrac{1}{2} PQ :: AP : \tfrac{1}{2} AC$$

and

$$KN : AC :: AB : CK$$

$\therefore KLON : DTV :: AP : AC$

:: vel. of the body at any time : the greatest vel.

Hence the increments of the areas $\propto$ velocity $\propto$ spaces described.

$\therefore$ by composition the whole $ABNK$: sector ATD :: space described to acquire any velocity : space described in a non-resisting medium for the same time.

21. COR. 2. In the same way, if the body ascend, the space described till the velocity $= Ap$: space through which a body would move :: $ABnk : ADt$.

22. COR. 3. Also, the velocity of a body falling for the time ATD : velocity which a body would acquire in a non-resisting medium in the same time :: $\triangle ADP$: sector TDA; for since the force is constant,

the velocity in a non-resisting medium $\propto$ time, and the force in a resisting medium $\propto$ A P $\propto \triangle$ A D P.

23. COR. 4. In the same way, the velocity in the ascent : velocity with which a body should move, to lose its whole motion in the same time : : $\triangle$ A p D : sector A t D : : A p : arc A t.

For let A Y be any other velocity acquired in a non-resisting medium in the same time with A P.

$\therefore$ A P : A C : : A P D : this area

and

A P : A C : : A P D : A C D.

Therefore the area which represents the time of acquiring the greatest velocity in a non-resisting medium = A C D.

In the same way, let A y be velocity lost in a non-resisting medium in the same time as A p in a resisting medium.

$\therefore$ A p : A y : : $\triangle$ A p D : area which represents the time of losing the velocity A p.

$\therefore$ time of losing the velocity A y = $\triangle$ A p D.

24. COR. 5. Hence the time in which a falling body would acquire the velocity A P : time in which, in a non-resisting medium, it would acquire the greatest velocity : : sector A D T : $\triangle$ C A D.

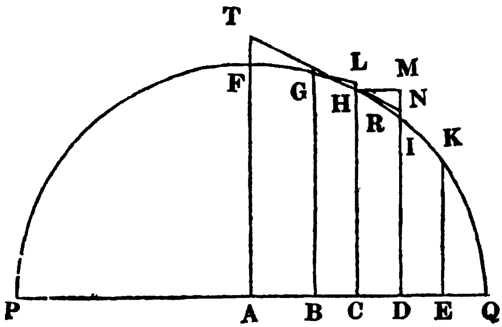
Also the time in which it would lose the velocity A p : time in which, in a non-resisting medium, it would lose the same velocity : : arc A t : tangent A p.

25. COR. 6. Hence the time being given, the space described in ascent or descent may be known, for the greatest velocity which the body can acquire is constant, therefore the time in which a body falling in a non-resisting medium, would acquire that velocity is also known. Then the sector A D T or A D t : $\triangle$ A D C : : given time : time just found; therefore the velocity A P is known or A p.

Then the area A B N K or A B n k : A D T or A D t : : space sought for : space which the body would describe uniformly with its greatest velocity.

26. COR. 7. Hence *vice versâ*, if the space be given, the time will be known.

27. Prop. X. Let P F Q be the curve meeting the plane P Q. Let



G, H, I, K be the points in the curve, draw the ordinates; let B C = C D = D E, &c.

Draw H N, G L tangents at H and G, meeting the ordinates produced in L and N, complete the parallelogram C H M D. Then the times $\propto \sqrt{LH}$ and $\sqrt{NI}$, and the velocities $\propto$ G H and H I, and the times $\propto$; let T and t = times, and the velocities $\propto \frac{GH}{T}$ and $\frac{HI}{t}$, therefore the decrement of the velocity arising from the retardation of resistance and the acceleration of gravity $\propto \frac{GH}{T} - \frac{HT}{t}$, also the accelerating force of gravity would cause a body to describe 2 I N in the same time, therefore the increment of the velocity from G $= \frac{2NI}{t}$, again the arc is increased by the space $= HI - HN = RI = \frac{MI \times NI}{HI}$, therefore the decrement from the resistance alone $= \frac{GH}{T} - \frac{HI}{t} + \frac{2MI \times NI}{t \times HI}$, $\therefore$ resistance : gravity :: $\frac{GH \times t}{T} - HI + \frac{2MI \times NI}{HI} : 2NI$.

Again, let

$$AB, CD, CE, \&c. \text{ be } -o + o, 2o, 3o, \&c.$$
$$CH = P$$

and

$$MI = Qo + Ro^2 + So^3 + \&c.$$
$$\therefore DI = P - Qo + \&c.$$
$$EK = P - 2Qo - 4Ro^2 - \&c.$$
$$BG = P + Qo + \&c.$$

$(BG - CH)^2 + BC^2 (= GH^2) = o^2 + Q^2 o^2 + 3QRo^3 + \&c.$

$$\therefore GH^2 = \overline{1+Q^2} \times o^2 + 3QRo^3,$$

$$\therefore GH = \sqrt{1+Q^2} \times o + \frac{QRo^2}{\sqrt{1+Q^2}}$$

and

$$HT = o\sqrt{1+Q^2} + \frac{QRo^2}{\sqrt{1+Q^2}}.$$

Subtract from C H $\frac{1}{2}$ the sum G B and D I, and Ro^2 and $Ro^2 + 3So^3$ will be the remainder, equal to the sagittæ of the arcs, and which are proportional to L H and N I, and therefore, in the subtracted number of the times,

$$\therefore \frac{t}{T} \propto \sqrt{\frac{R + 3So}{R}} \propto \frac{R + \frac{3}{2}So}{2R} \propto 1 + \frac{3So}{2R}.$$

$$\therefore \frac{GH \times t}{T} = \overline{o\sqrt{1+Q^2} + \frac{QRo^2}{\sqrt{1+Q^2}}} \times \overline{1 + \frac{3So^2}{2R}}$$

$$= o.\sqrt{1+Q^2} + \frac{QRo^2}{\sqrt{1+Q^2}} + \frac{3So^3\sqrt{1+Q^2}}{2R} + \frac{3So}{2R} \times \frac{QRo^2}{\sqrt{1+Q^2}}$$

$$HI = o.\sqrt{1+Q^2} + \frac{QRo^2}{\sqrt{1+Q^2}}$$

$$\frac{MI \times NI}{HI} = \frac{Ro^2 \times Qo + Ro^2 + \&c.}{o.\sqrt{1+Q^2} + \frac{QRo^2}{\sqrt{1+Q^2}}}$$

$$\therefore \text{resistance} : \text{gravity} :: \frac{GH \times t}{T} - HI + \frac{2MI \times NI}{HI} : 2NI$$

$$:: \frac{3So^2\sqrt{1+Q^2}}{2R} : 2Ro^2$$

$$:: 3S\sqrt{1+Q^2} : 4R^2.$$

The velocity is equal to that in the parabola whose diameter $=$ H C, and the lat. rect. $= \frac{HN^2}{TN}$ or $\frac{1+Q^2}{R}$. The resistance $\propto$ density $\times V^2$, therefore the density $\propto \frac{\text{resistance}}{V^2} \propto \frac{3S\sqrt{1+Q^2}}{4R^2}$ directly $\propto \frac{R}{1+Q^2}$ directly $\propto \frac{S}{R\sqrt{1+Q^2}}$.

28. Ex. 1. Let it be a circular arc, C H $= e$, A Q $= n$, A C $= a$, C D $= o$,

$\therefore DI^2 = n^2 - (a+o)^2 = n^2 - a^2 - 2ao - o^2 = e^2 - 2ao - o^2,$

$$\therefore DI = e - \frac{ao}{e} - \frac{n^2 o^2}{2e^3} - \frac{a n^2 o^3}{2e^5},$$

and therefore

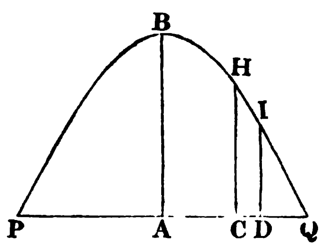
$$P = e,\ Q = \frac{a}{e},\ R = \frac{n^2}{2e^3},\ S = \frac{a n^2}{2e^5},$$

$$\therefore \text{density} \propto \frac{S}{R\sqrt{1+Q^2}} \propto \frac{a n^2}{2e^5} \times \frac{2e^3}{n^2} \times \frac{e}{n},$$

$$\propto \frac{a}{ne} \propto \frac{a}{e} \propto \frac{\sin.}{\cos.} \propto \text{tangent.}$$

The resistance : gravity : : $\frac{3 a n^2}{2 e^5} \times \frac{n}{e} : \frac{n^4}{e^5} : : 3a : 2n.$

29. Ex. 2. Of the hyperbola.



$$PI \times b = PD^2,$$

$$\therefore \text{put } PC = a,\ CD = o,\ QP = c,$$

$$\therefore \overline{a+o} \times \overline{c-a-o} = ac - a^2 - 2ao + co - o^2$$

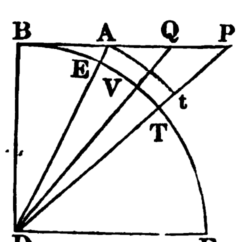
$$\therefore DI = \frac{ac-a^2}{b} - \frac{2a+c}{b} . o - \frac{o^2}{b},$$

and since there is no fourth term,

$$S = 0,$$

$$\therefore \text{draw } y = 0.$$

30. Prop. XIII. Suppose the resistance to $\propto V + V^2$.



Case 1. Suppose the body to ascend; with the center D and rad. D B,

describe the quadrant B T F; draw B P an indefi nite line perpendicular to B D, and parallel to D F. Let A P represent the velocity; join D P, D A, and draw D Q near D P.

$\therefore$ resistance $\propto AP^2 + 2BA \times AP$, suppose gravity $\propto DA^2$,

$\therefore$ decrement of V $\propto$ gravity + resistance $\propto AD^2 + AP^2 + 2BA \times AP$.

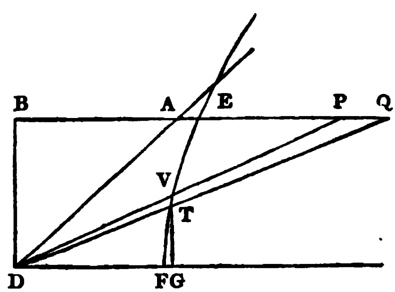
$$\propto DP^2$$

$$DPQ (\propto PQ) : DTV :: DP^2 : DT^2,$$

$$\therefore DTV \propto DT^2 \propto 1,$$

therefore the whole sector E T D, is proportional to the time.

Case 2. Suppose the force of gravity proportional to a less quantity than DA^2, draw B D perpendicular to B P, and let the force of gravity



$\propto AB^2 - BD^2$. Draw D F parallel to P B and = D B and with the center D — $\frac{1}{2}$ axis-major = $\frac{1}{2}$ axis-minor = D B, describe a hyperbola from the vertex F, cutting A D produced in E, and D P, D Q in T, V.

Now since the body is supposed to ascend.

The decrement of the velocity $\propto AP^2 + 2AB \times AP + AB^2 - BD^2 \propto BP^2 - BD^2 (BP^2 = AP^2 + AB^2 + 2AB \times BP)$.

Also, $DTV : DPQ :: DT^2 : DP^2$ (by similar triangles)

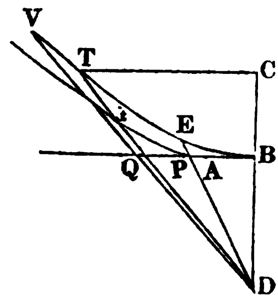
$:: TG^2 : BD^2$ (T G perpendicular to G)

$:: DF^2 : PB^2 - DB^2$.

Now D P Q $\propto$ decrement of velocity $\propto PB^2 - DB^2$,

$\therefore DTV \propto DF^2 \propto 1 \propto$ increment of the time, since the time flows uniformly.

Case 3. If the body descend; let gravity $\propto BD^2 - AB^2$.



With center D and vertex B, describe the rectangular hyperbola B T V, cutting the lines D A, D P, D Q produced in E, T, V.

The increment of $V \propto BD^2 - AB^2 - 2AB \times AP - AP^2$

$$\propto BD^2 - (AB + AP)^2 \propto BD^2 - BP^2$$

$$DTV : DPQ\,(\propto PQ) :: DT^2 : DP^2$$

$$:: GT^2 : BP^2 :: GD^2 - BD^2 : BP^2$$

$$:: GD^2 : BD^2 :: BD^2 : BD^2 - BP^2,$$

$$\therefore DTV \propto BD^2 \propto 1,$$

$\therefore$ the whole sector E D T $\propto$ time.

31. Cor. With the center C and distance D A describe an arc similar to B T.

Then the velocity A P : the velocity which in the time E D t a body would lose or acquire in a non-resisting medium : : $\triangle$ D A P : sector A D t.

For V in a non-resisting medium $\propto$ time.

32. In the case of the ascent,

Let the force of gravity $\propto 1$. Resistance $\propto 2av + v^2$

$$\therefore dv \propto 1 + 2av + v^2$$

$$\therefore \frac{dv}{1 + 2av + v^2} \propto \text{time.}$$

$\therefore$ by Demoivre's first formula,

f. or time $= 0$

when

$$f.\ \frac{dv}{1 + 2av + v^2} = \frac{1}{g^2} \times \text{cir. arc. rad.} = g \text{ and}$$

$$\text{tangent} = v + a$$

The whole time $\therefore$ when $v = 0 = \frac{1}{g^2} \times$ cir. arc rad. $= g$
and tangent $= a + C$.

$\therefore$ cor^t. time $= \frac{1}{g^2} \times$ cir. arc rad. $= g$ and tangent $v + a$ — cir. arc rad. $= g$ and tangent a.

$\therefore$ the time of ascent $=$ sector E D T — $g^2 = 1 - a^2$.

33. In the case of descent,

$$d\,v \propto 1 - 2\,a\,v - v^2$$

let

$$v + a = x$$

$$\therefore d\,v = d\,x$$

$$\therefore v^2 + 2\,a\,v + a^2 = x^2$$

$$\therefore 1 + a^2 - x^2 = 1 - 2\,a\,v - v^2$$

$$\therefore f. = \frac{1}{2g} \times \int \frac{g+x}{g-x} + C, \; (g^2 = 1 + a^2)$$

$$\text{Time} = 0, \quad v = 0,$$

$$\therefore x = a,$$

$$\therefore o = \frac{1}{2g} \times \int \frac{g+a}{g-a} + C$$

$$\therefore \text{Cor}^t.\ \text{time} = \frac{1}{2g} \times \int \frac{g+x}{g-x} - \int \frac{g+a}{g-a}.$$

34. PROP. XIV. Take A C proportional to gravity, and A K to the resistance on contrary sides if the body ascend, and *vice versâ.*

Between the asymptotes describe a hyperbola, &c. &c.

Draw A b perpendicular to C A, and

$$A\,b : D\,B :: D\,B^2 : 4\,B\,A \times A\,C.$$

The area A b N K increases or decreases in arithmetic progression it the forces be taken in geometric progression.

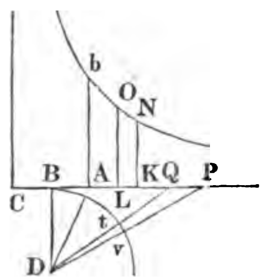
Now

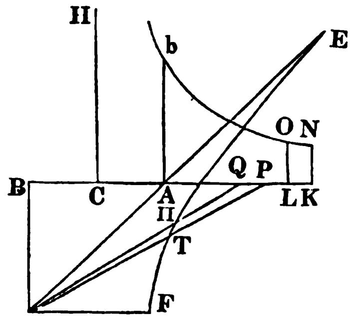
$$A\,K \propto \text{resistance} \propto 2\,B\,A\,P + A\,P^2.$$

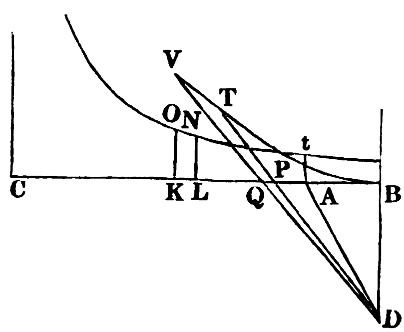
Let

$$A\,K = \frac{2\,B\,A\,P + A\,P^2}{Z}$$

$$\therefore K\,L = \frac{2\,B\,A \times P\,Q + 2\,A\,P \times P\,Q}{Z},$$







$$KL = \frac{2BPQ}{Z}$$

$$\therefore KLON = \frac{2BP \times PQ}{Z} \times LO.$$

Now

$$Ab : LO :: CK : CA$$

$$DB : Ab :: 4BA \times CA : DB^2$$

$$\therefore LO = \frac{BD^3}{4BA \times CK}$$

$$\therefore KLON = \frac{2PB \times PQ \times BD^3}{4BA \times CK \times Z}.$$

Case 1. Suppose the body to ascend.

$$\text{gravity} \propto AB^2 + BD^2 = \frac{AB^2 + BD}{Z}$$

$$AK = \frac{AP^2 + 2BAP}{Z}$$

$$\therefore DP^2 = CK \times Z.$$

$$\therefore DT^2 : DP^2 :: DB^2 : CK \times Z$$

and in the other two cases the same result will obtain.

Make

$$DTV = DB \times m.$$

$$\therefore DB \times m : \tfrac{1}{2} DB \times PQ :: DB^2 : CK \times Z$$

$$\therefore BD^3 \times PQ = 2BD \times m \times CK \times Z.$$

$$\therefore AbNK = \frac{BP \times BD \times m}{AB}$$

$$\therefore AbNK - DTV = \frac{\overline{BP - AB} \times BD \times m}{AB} \propto AP \cdot \propto \text{velocity}.$$

$\therefore$ it will represent the space.

SECTION IV.

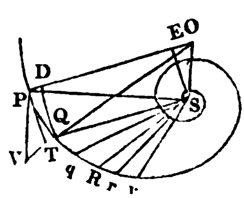
35. PROP. XV. LEMMA. The

$$\angle OPQ = \text{a rectangle} = \angle OQR$$

and

$$\angle SPQ = \angle \text{ of the spiral} = \angle SQR$$

$$\therefore \angle OPS = \angle OQS.$$



$\therefore$ the circle which passes through the points P, S, O, also passes through Q. Also when Q coincides with P, this $\frac{\text{circle}}{2}$ touches the spiral.

$$\therefore \angle PSO \quad \angle \text{ in a } \frac{\text{circle}}{2} \text{ whose diameter} = PO.$$

Also

$$T Q : P Q :: P Q : 2 P S.$$

$$\therefore P Q^2 = 2 P S \times T O$$

which also follows from the general property of every curve.

$$P Q^2 = P V \times Q R.$$

$$\therefore Q R = \frac{P Q^2}{2 W}.$$

36. Hence the resistance $\propto$ density $\times$ square of the velocity.

37. Density $\propto \frac{1}{\text{distance}}$, centripetal force $\propto$ density^2 $\propto \frac{1}{\text{distance}^2}$.

Then produce S Q to V so that S V = S P, and let P Q be an arc described in a small time, P R described in twice that time, $\therefore$ the decrements of the arcs from what would be described in a non-resisting medium $\propto T^2$.

$\therefore$ decrement of the arc P Q = $\frac{1}{4}$ decrement of the arc P R

$\therefore$ decrement of the arc P Q = $\frac{1}{2}$ R r (if Q S r = area P S Q).

For let P q, q v be arcs described (in the same time as P Q, Q R) in a non-resisting medium,

$$\begin{aligned} P S q - P S Q = Q S q &= q S v - Q S r \\ &= r S v - Q S q \end{aligned}$$

$$\therefore 2 Q S q = r S v$$

$\therefore$ if S T ultimately = S t be the perpendicular on the tangents

$$S T \times Q q = \tfrac{1}{2} S t \times r v$$

$$\therefore 2 Q q = r v$$

and

$$R v = 4 Q q.$$

$$\therefore 2 Q q = R r.$$

Hence

$$\text{Resistance} : \text{centripetal force} :: \tfrac{1}{2} R r : T Q,$$

Also

$$T Q \times S P^2 \propto \text{time}^2, \text{ (Newt. Sect. II.)}$$

$$\therefore P Q^2 \times S P \propto \text{time}^2$$

$$\therefore \text{time} \propto P Q \times \sqrt{S P}$$

$$\therefore V \propto \frac{P Q}{P Q \times \sqrt{S P}} \propto \frac{1}{\sqrt{S P}}$$

also

$$V \text{ at } Q \propto \frac{1}{\sqrt{S Q}}$$

$$PQ : QR :: \sqrt{SQ} : \sqrt{SP}$$
$$:: SQ : \sqrt{SQ \times SP}$$
$$PQ : Qr :: SQ : SP$$

since the areas are equal, and the angles at P and Q are equal.

$$\therefore PQ : Rr :: SQ : SP - \sqrt{SQ \times SP}$$
$$:: SQ : \tfrac{1}{2} VQ$$

For

$$SQ = SP - VQ$$
$$\therefore SQ \times SP = SP^2 - VQ \times SP$$
$$\therefore \sqrt{SQ \times SP} = SP - \tfrac{1}{2} VQ - \frac{VQ^2}{8\,SP} - \&c.$$
$$\therefore \tfrac{1}{2} VQ \text{ ultimately} = SP - \sqrt{SP \times SQ}$$
$$\text{Resistance} \propto \frac{\text{decrement of } V}{\text{time}^2} \propto \frac{Rr}{PQ^2 \times SP}$$
$$\propto \frac{\tfrac{1}{2} VQ}{PQ \times SQ \times SP}$$
$$\tfrac{1}{2} VQ : PQ :: \tfrac{1}{2} OS : PO$$

and

$$SQ = SP \propto \frac{\tfrac{1}{2} OS}{OP \times SP^2}$$

$$\therefore \text{density} \times \text{square of the velocity} \propto \text{resistance} \propto \frac{OS}{OP \times SP^2}$$
$$\therefore \text{density} \propto \frac{OS}{OP \times SP}$$

and in the logarithmic spiral $\frac{OS}{OP}$ is constant.

$$\therefore \text{density} \propto \frac{1}{SP}. \quad \text{Q. e. d.}$$

38. COR. 1. V in spiral = V in the circle in a non-resisting medium at the same distance.

39. COR. 3. Resistance : centripetal force :: $\tfrac{1}{2} Rr : TQ$

$$:: \frac{\tfrac{1}{2} VQ \times PQ}{SQ} : \frac{\tfrac{1}{2} PQ^2}{SP}$$
$$:: \tfrac{1}{2} VQ : PQ$$
$$:: \tfrac{1}{2} OS : OP.$$

$\therefore$ the ratio of resistance to the centripetal force is known if the spiral be given, and *vice versâ*.

40. COR. 4. If the resistance exceed $\tfrac{1}{2}$ the centripetal force, the body cannot move in this spiral. For if the resistance equal $\tfrac{1}{2}$ the centripetal

force, O S = O P, $\therefore$ the body will descend to the center in a straight line P S.

V of descent in a straight line : V in a non-resisting medium of descent in an evanescent parabola : : 1 : $\sqrt{2}$; for V in the spiral = V in the circle at the same distance, V in the parabola = V in the circle at $\frac{1}{2}$ distance.

Hence since time $\propto \frac{1}{V}$,

time of descent in the 1st case : that in 2d : : $\sqrt{2}$: 1.

41. Cor. 5. V in the spiral P Q R = V in the line P S at the same distance. Also

$$PQR : PS \text{ in a given ratio} :: PS : PT :: OP : OS$$

$\therefore$ time of descending P Q R : that of P S : : O P : O S.*

Length of the spiral = T P = sector of the $\angle$ T P S.

$$a : b :: b : c :: c : d :: d : e$$

$$a + b + c + \&c. : b + c + d + \&c. :: a : b$$

$$\therefore a + b + c + \&c. : a :: a : \overline{a - b}.$$

42. Cor. 6. If with the center S and any two given radii, two circles be described, the number of revolutions which the body makes between the two circumferences in the different spirals $\propto$ tangent of the angle of the spiral $\propto \frac{PS}{OS}$.

The time of describing the revolution : time down the difference of the radii : : length of the revolution : that difference.

2d $\propto$ 4th,

$\therefore$ time $\propto$ length of the revolution $\propto$ secant of the angle of the spiral

$$\propto \frac{OP}{OS}.$$

* $pq : pt :: Sp : Sy$

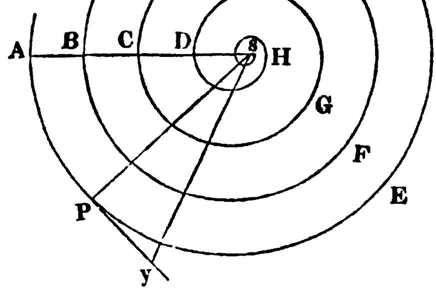
or

$$dw : \frac{p\,dx}{\sqrt{r^2 - p^2}} :: x : p.$$

$$\therefore dw = \frac{x\,dx}{\sqrt{r^2 - p^2}}$$

$$\therefore w = \frac{x^2}{2\sqrt{r^2 - p^2}}.$$

43. COR. 7. Suppose a body to revolve as in the proposition, and to cut



the radius in the points A, B, C, D, the intersections by the nature of the spiral are in continued proportion.

$$\text{Times of revolution} \propto \frac{\text{perimeters described}}{V}$$

$$\text{and velocity} \propto \frac{1}{\sqrt{\text{distance}}}$$

$$\propto AS^{\frac{3}{2}},\ BS^{\frac{3}{2}},\ CS^{\frac{3}{2}},$$

$$\therefore \text{ the whole time : time of one revolution} :: AS^{\frac{3}{2}} + BS^{\frac{3}{2}} + \&c. : AS^{\frac{3}{2}}$$

$$:: AS^{\frac{3}{2}} : AS^{\frac{3}{2}} - BS^{\frac{3}{2}}.$$

44. PROP. XVI. Suppose the centripetal force $\propto \dfrac{1}{SP^{n+1}}$,

$$\text{time} \propto PQ \times SP^{\frac{n}{2}}$$

$$\text{and velocity} \propto \frac{1}{SP^{\frac{n}{2}}}$$

$$PQ : QR :: SQ^{\frac{n}{2}} : SP^{\frac{n}{2}}$$
$$Qr : PQ :: SP : SQ$$

$$Qr : QR :: SQ^{\frac{n}{2}-1} : SP^{\frac{n}{2}-1}$$

$$\therefore Qr : Rr :: SQ^{\frac{n}{2}-1} : SQ^{\frac{n}{2}-1} - SP^{\frac{n}{2}-1}$$
$$:: SQ : \overline{1 - \tfrac{1}{2}n} \,.\, VQ.$$

For

$$SP = SQ + VQ,$$

$$\therefore SP^{\frac{n}{2}-1} = SQ^{\frac{n}{2}-1} + \overline{\frac{n}{2}-1}.\ VQ \times SQ^{\frac{n}{2}-2} + \&c.$$

$$\therefore SQ^{\frac{n}{2}-1} - SP^{\frac{n}{2}-1} = \overline{1-\frac{n}{2}} \times VQ \times SQ^{\frac{n}{2}-2}.$$

Then as before it may be proved, if the spiral be given, that the density $\propto \frac{1}{SP}$. Q. e. d.

45. Cor. 1.

Resistance : centripetal force : : $\overline{1-\frac{1}{2}n}\,.\,OS : OP$,

for the resistance : centripetal force : : $\frac{1}{2}$ R r : T Q

$$: : \frac{\left(1-\frac{n}{2}\right) \times VQ \times PQ}{2SQ} : \frac{PQ^2}{2SP}$$

$$: : \overline{1-\frac{n}{2}} \times VQ : PQ$$

$$: : \overline{1-\frac{n}{2}} \times OS : OP.$$

46. Cor. 2. If $n + 1 = 3$, $1 - \frac{n}{2} = 0$,

$\therefore$ resistance $= 0$.

Cor. 3. If $n + 1$ be greater than 3, the resistance is propelling.

SECTION VI.

47. Prop. XXIV. The distances of any bodies' centers of oscillation from the axis of motion being the same, the quantities of matter $\propto$ weight $\times$ squares of the times of oscillation *in vacuo*.

For the velocity generated $\propto \frac{\text{force} \times \text{time}}{\text{quantities of matter}}$. Force on bodies at equal distances from the lowest points $\propto$ weights, times of describing corresponding parts of the motion $\propto$ whole time of oscillation,

$$\therefore \text{quantities of matter} \propto \frac{\text{force} \times \text{time of oscil.}}{\text{velocities}}$$

$$\propto \text{weights} \times \text{squares of the times},$$

since the velocities generated $\propto \frac{1}{\text{times}}$ for equal spaces.

48. Cor. 1. Hence the times being the same, the quantities of matter $\propto$ weights.

Cor. 2. If the weights be the same, the quantities of matter $\propto$ time^2.

Cor. 3. If the quantities of matter be the same, the weights $\propto \frac{1}{\text{time}^2}$.

49. COR. 4. Generally the accelerating force $\propto \frac{\text{Weights}}{\text{quantities}}$ of matter,

$$\text{and } L \propto T\,T^2,$$

$$\therefore \dot{L} \propto \frac{W \times T^2}{Q},$$

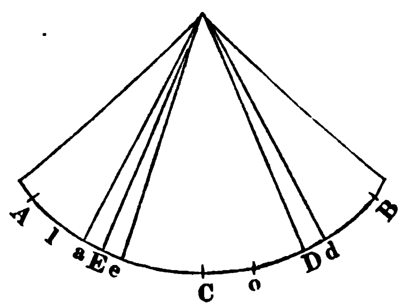
$$\therefore Q \propto \frac{W \times T^2}{L},$$

$\therefore$ if W and Q be given $L \propto T^2$.

If T and Q be given $L \propto W$.

50. COR. 5. generally the quantity of matter $\propto \frac{\text{weight} \times \text{time}^2 \text{ of oscillation}}{\text{length}}$.

51 PROP. XXV. Let A B be the arc which a body would describe in a



non-resisting medium in any time. Then the accelerating force at any point D $\propto$ C D; let C D represent it, and since the resistance $\propto$ time, it may be represented by the arc C o.

$\therefore$ the accelerating force in a resisting medium of any body d, = o d. Take

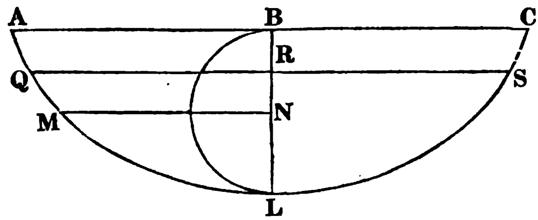
o d : C D : : o B : C B.

Therefore at the beginning of motion, the accelerating force will be in this ratio, $\therefore$ the initial velocities and spaces described will be in the same ratio, $\therefore$ the spaces to be described will also be in the same ratio, and vanish together, $\therefore$ the bodies will arrive at the same time at the points C and o.

In the same way when the bodies ascend, it may be proved that they will arrive at their highest points at the same time. $\therefore$ If A B : a B in the ratio C B : o B, the oscillations in a non-resisting and resisting medium will be isochronous. Q. e. d.

Cor. The greatest velocity in a resisting medium is at the point o.

The expression for the $\frac{1}{2}$ time of an oscillation in vacuo, or time of descent down to the lowest point $\propto$ quadrant whose radius $= 1$. Now



suppose the body to move in a resisting medium when the resistance : force of gravity : : r : 1.

Then $v\,d\,v = -g\,F\,d\,x + g\,r\,d\,z = -g\,d^2\,x + g\,r\,d\,z$. Now by a property of the cycloid, if $\frac{a}{2}$ be the axis, $d\,x : d\,z :: x : \frac{z}{2} :: z : a$,

$$\therefore d\,x = \frac{z\,d\,z}{a},$$

$$\therefore v\,d\,v = -\frac{g}{a} \times z\,d\,z + g\,r\,d\,z - \frac{v^2}{2}$$

$$= -\frac{g}{a} \times z^2 + g\,r\,z,$$

$$\therefore v^2 = \frac{-g}{a} \times z^2 + 2\,g\,r\,z + C.$$

Now

$$z = d,\ v = o,$$

$$\therefore v^2 = \frac{g}{a} \times \overline{d^2 - z^2} - 2\,g\,r \times \overline{d - z}$$

$$= \frac{g}{a} \times \overline{d^2 - 4\,a\,r\,d + 2\,a\,d\,r\,z - z^2},$$

$$\therefore v = \int \frac{g}{a} \times \sqrt{d^2 - 2\,a\,r\,d + 2\,a\,r\,z - z^2},$$

$$\therefore d\,t = \frac{-dz}{v} = \int \frac{a}{g} \times \frac{-d\,z}{\sqrt{d^2 - 2\,a\,r\,d + 2\,a\,r\,z - z^2}}.$$

Assume

$$z - a\,r = y,$$

$$\therefore z^2 - 2\,a\,r\,z + a^2\,r^2 = y^2,$$

$$\therefore 2\,a\,r\,z - z^2 = a^2\,v^2 - y^2,$$

$$d^2 - 2\,a\,r\,d + 2\,a\,r\,z - z^2 = (d - a\,r)^2 - y^2 = (b^2 - y^2.)$$

and

$$d z = d y$$

$$\therefore d t = \int \frac{a}{g} \times \frac{-d y}{\sqrt{b^2 - y^2}}$$

$$\therefore t = \int \frac{a}{g} \times \text{circular arc, radius} = 1,$$

and

$$\cos. = \frac{z - a r}{d - a r} + C \text{ and } C = o.$$

$\therefore$ the whole time of descent to the lowest point $= \int \frac{a}{g} \times$ circular arc whose cos. $= \frac{-a r}{d - a r}$, $\therefore$ time in vacuo : time in resisting medium :: quadrant : arc whose cos. $= \frac{-a r}{d - a r}$.

COR. 1. Time of descent to the point of greatest acceleration is constant, for in that case $z = a r$,

$$\therefore t = \int \frac{a}{g} \times \text{quadrant, for } d v = 0,$$

$$\therefore v d v = 0,$$

$$\therefore - g z d z + g a r z = 0,$$

$$\therefore z = a r,$$

$$\therefore z : r :: a : 1.$$

COR. 2. To find the excess of arc in descent above that in ascent.

$$v d v = + g T d x + g r d z,$$

$$\therefore v d v = - \frac{g z d z}{a} - g r d z$$

$$\therefore \frac{v^2}{2} = - \frac{m z^2}{a} - g r z + C,$$

$$\therefore v^2 = \frac{g}{a} (d^2 - z^2) - (z - d) \times 2 a r$$

$$= \frac{g}{a} \times (d^2 - 2 a r d) - (2 a r z - z^2)$$

which when the body arrives to the highest point $= 0$,

$$d^2 - 2 a r d - 2 a r z - z^2 = 0,$$

$$\therefore d^2 - 2 a r d = z^2 + 2 a r z,$$

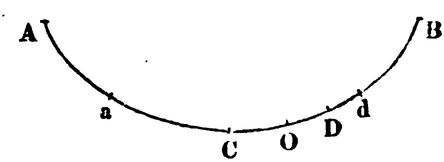
$$\therefore z + a r = d - a r,$$

$$\therefore z = d - 2 a r,$$

$$\therefore d - z = 2 a r.$$

52. PROP. XXVI. Since V $\propto$ arc, and resistance $\propto$ V, resistance $\propto$ arc.
$\therefore$ Accelerating force in the resisting medium $\propto$ arcs.
Also the increments or decrements of V $\propto$ accelerating force.
$\therefore$ the V will always $\propto$ arc.
But in the beginning of the motion, the forces which $\propto$ arcs will generate velocities which are proportional to the arcs to be described. $\therefore$ the velocities will always $\propto$ arcs to be described.
$\therefore$ the times of oscillation will be constant.

53. PROP. XXVIII. Let C B be the arc described in the descent, C a in the ascent.

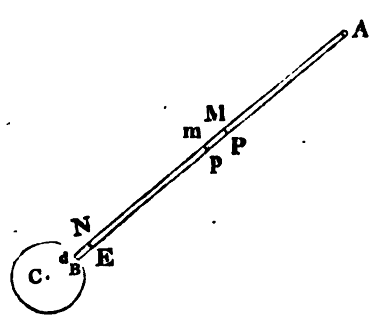


$\therefore$ A a = the difference (if A C = C B)
Force of gravity at D : resistance : : C D : C O.

$$C A = C B$$
$$O a = O B$$
$$\therefore C A - O a \text{ or } A a - C O = C B - O B = C O$$
$$\therefore C O = \tfrac{1}{2} A a$$

$\therefore$ Force of gravity at D : resistance : : C D : $\frac{1}{2}$ A a
$\therefore$ At the beginning of the motion,
Force of gravity : resistance : : 2 C B : A a
: : 2 length of pendulum : A a.

54. PROB. To find the resistance on a thread of a sensible thickness.



Resistance $\propto$ $V^2 \times D^2$ of suspended globe.
$\therefore$ resistance on the whole thread : resistance on the globe C : :

$$:: 2a^3b^2.(a-b)^2 : a^3r^2c^2 - r^2c^2.(a-2b)^3, \qquad c = a + r.$$
$$:: a^3b^2.(a-b)^2 : 3a^2r^2c^2b - bab^2r^2c^2 + 4b^3r^2c^2,$$
$$:: a^3b..(a-b)^2 : 3a^2r^2c^2 - babr^2c^2 + 4b^2r^2c^2,$$

$\therefore$ resistance on the thread : whole resistance

$$:: a^3b.(a-b)^2 : r^2c^2.(3a^2 - bab + 4b^2).$$

COR. If the thickness (b) be small when compared with the length (a)

$$3a^2 - bab + 4b^2 = 3a^2 - bab + 3b^2 \text{(nearly)} = 3.(a-b)^2.$$

$\therefore$ Resistance on the whole thread : resistance on the globe

$$:: a^3b : 3r^2c^2$$

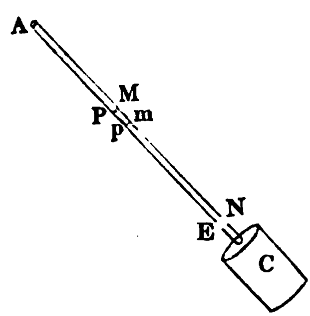
and

Resistance on the thread : whole resistance to the pendulum

$$:: a^3b : a^3b + 3r^2c^2.$$

Suppose, instead of a globe, a cylinder be suspended whose ax. $= 2r$. Now by differentials

the resistance on the circumference : resistance on the base : : 2 : 3.



By composition the resistance to the cylinder : resistance on the square $= 2r :: 2 : 3$.

$$\text{Resistance } \propto x^2x',$$
$$\therefore \text{resistance } \propto x^3,$$
$$\therefore \text{resistance to the whole thread } \propto x^3.$$

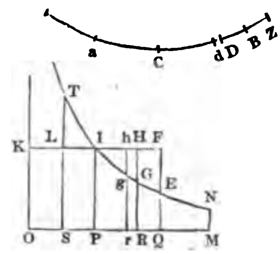
Resistance on A E $\propto (a-2b)^3$ if $2b = $ E D.

$\therefore$ Resistance on the thread : resistance of the globe

$$:: 16.a^3b^2.(a-b)^2 : 3p.\overline{a^3-(a-2b)^3} \times r^2.(a+r)^2.$$

55. PROP. XXIX. B a is the whole arc of oscillation. In the line O Q take four points S, P, Q, R, so that if O K, S T, P I, Q E be erected

perpendiculars to O Q meeting a rectangular hyperbola between the asymptotes O Q, O K in T, I, G, E, and through I, K F be drawn



parallel to O Q, meeting Q E produced in F. The area P I E Q may be : area P I S T : : C B : C a. Also I E F : I L T : : O R : O S.

Draw M N perpendicular to O Q meeting the hyperbola in N, so that P L M N may be proportional to C Z, and P I G R to C D.

Then the resistance : gravity : : $\frac{OR}{OQ} \times$ T E F — I G H : P I N M.

Now since the force $\propto$ distance, the arcs and forces are as the hyperbolic areas. $\therefore$ D d is proportional to R r G g.

Now by taking the differentials the increment of $\left(\frac{OR}{OQ} TEF - IGH\right)$ $= GHgh - \frac{Rr.IEF}{OQ} : Rr \times GR :: HG - \frac{IEF}{OQ} : GR :: OR \times HG - IEZ \times \frac{OR}{OQ} : OP \times PI - (OR \times HG = OR \times HR - OP \times PI = PIHR = PIRG + IGH) :: PIRG + IGH - \frac{OR}{OQ} \times IEF : OP \times PI.$

Now if $Y = \frac{OR}{OQ} \times IEF - IGH$, the increment $Y \propto PIGR - Y$.

Let V = the whole from gravity. $\therefore$ V — R = actual accelerating force. $\therefore$ Increment of the velocity $\propto \overline{V - R} \times$ increment of the time. As the resistance $\propto V^2$ the increment of resistance $\propto$ V $\times$ increment of the velocity, and the velocity $\propto \frac{\text{increment of the space}}{\text{increment of the time}}$. $\therefore$ Increment of resistance $\propto$ V — R if the space be given, $\propto$ P I G R — Z, if Z be the area which represents the resistance R e.

Since the increment Y $\propto$ P I G R — Y, and the increment of Z

$\propto$ P I G R — Z. If Y and Z be equal at the beginning of the motion and begin at the same time by the addition of equal increments, they will still remain equal, and vanish at the same time.

Now both Z and Y begin and end when resistance $= 0$, i. e. when

$$\frac{OR}{OQ} \cdot IEF - IGH = 0$$

or

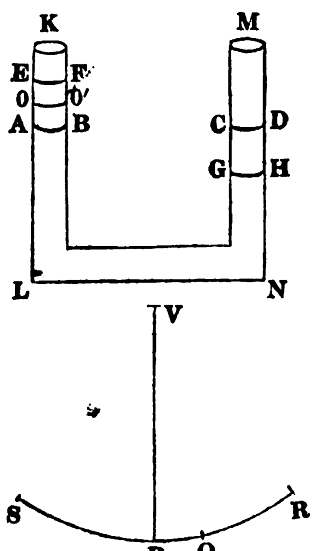
$$\frac{ILT}{OS} \times OR - IGH = 0.$$

$$\therefore \frac{OR \times IEF}{OQ} - IGH = Z$$

$$\therefore \text{Resistance} : \text{gravity} :: \frac{OR}{OQ} \cdot IEF - IGH : PMNI.$$

SECTION VIII.

56. PROP. XLIV. The friction not being considered, suppose the mean



altitude of the water in the two arms of the vessel to be A B, C D. Then when the water in the arm K L has ascended to E F, the water in the arm M N will descend to G H, and the moving force of the water equals the excess of the water in one arm above the water in the other, equals twice A E F B. Let V P be a pendulum, R S a cycloid $= \frac{1}{2}$ length of the canal, and P Q = A E. The accelerating force of the water : whole weight :: A E or P Q : P R.

Also, the accelerating force of P through the arc P Q : whole weight of P : : P Q : P R; therefore the accelerating force of the water and P $\propto$ the weights. Therefore if P equal the weight of the water in the canal, the vibration of the water in the canal will be similar and cotemporaneous with the oscillations of P in the cycloid.

COR. 1. Hence the vibrations of the water are isochronous.

COR. 2. If the length of the canal equal twice the length of the pendulum which oscillates in seconds; the vibrations will also be performed in seconds.

COR. 3. The time of a vibration will $\propto \sqrt{L}$.

Let the length $= L$, $A E = a$,

then the accelerating force : whole weight : : $2a : L$,

$$\therefore \text{accelerating force} = \frac{2a}{L};$$

$$\therefore \text{when the surface is at } 0, \text{ the accelerating force} = \frac{2 A 0}{L}.$$

$$\text{Put } E 0 = x,$$

$$A 0 = a - x,$$

$$\therefore \text{accelerating force} = \frac{2a - 2x}{L},$$

$$\therefore v\,d\,v = \frac{g \cdot \overline{2a\,dx - 2x\,dx}}{L},$$

$$\therefore v^2 = \frac{2g}{L} \times \overline{2ax - x^2},$$

$$\therefore v = \sqrt{\frac{2g}{L}} \times \sqrt{2ax - x^2}$$

$$d t = \frac{dx}{v} = \sqrt{\frac{L}{2ga^2}} \times \frac{a\,dx}{\sqrt{2ax - x^2}},$$

$\therefore t = \sqrt{\frac{L}{2ga^2}} \times$ cir. arc rad. $= a$, and vers. $= x$ + cor^n. and cor^n. $= 0$,

$$\because t = 0,\ x = 0,$$

$$\therefore \text{if } p = 3.\ 14159, \&c.$$

$$t = \sqrt{\frac{L}{2ga}} \times \frac{pa}{2}\ \{\text{when } (x) = (a)\} = \sqrt{\frac{L}{2g}} \times \frac{p}{2}$$

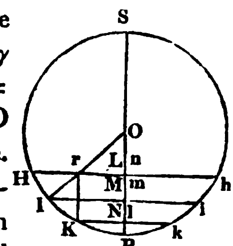
$\therefore$ time of one entire vibration $= p \times \sqrt{\frac{L}{2g}} =$ time of one entire vibration of a pendulum whose length $= \frac{L}{2}$.

57. COR. 1. Since the distance (a) above the quiescent surface does not enter into the expression. The time will be the same, whatever be the value of A E.

58. COR. 2. The greatest velocity is at $A = \sqrt{\frac{2g}{L}} \times a, \propto \frac{L}{L^{\frac{1}{2}}} \propto \sqrt{\frac{a^2}{l}}$

$$\propto \sqrt{\frac{AE^2}{L}}.$$

59. PROP. XLVII. Let E, F, G be three physical points in the line B C, which are equally distant; E e, F f, G g the spaces through which they move during the time of one vibration. Let ϵ, φ, γ be their place at any time. Make P S = E e, and bisect it in O, and with center O and radius O P = O S, describe a circle. Let the circumference of this circle represent the time of one vibration, so that in the time P H or P H S h, if H L or h l be drawn perpendicular to P S and E ϵ be taken = P L or P l, E ϵ may be found in E; suppose this the nature of the medium. Take in the circumference P H S h, the arcs H I, I K, h i, i k which may bear the same ratio to the circumference of the circle as E F or F G to B C. Draw I M, K N or i m, k n perpendicular to P S. Hence P I, or P H S i will represent the motion of F. and P K or P H S k that of G. E ϵ, F φ, G γ = P L, P M, P N or P l, P m, P n respectively.



Hence $\epsilon\gamma$ or E G + G γ — E ϵ = G E — L N = expansion at $\epsilon\gamma$; or = E G + l n.

∴ in going, expansion : mean expansion : : G E — L N : E G

In returning, ——— : ————— : : E G + l n : E G

Now join I O, and draw K r perpendicular to H L, H K r, I O M are similar triangles, since the ∠ K H r = ½ K O k = ½ I O i = ∠ I O P and ∠ at r and M = 90°,

∴ L N : K H : : I M : I O or O P, and by supposition K H : E G : : circumference P S L P : B C : : O P : V = radius of the circle whose circumference = B C.

∴ by composition L N : G E : : I M : V.

∴ expansion : mean expansion : : V — I M : V,

$\therefore$ elasticity : mean elasticity $:: \frac{1}{V - IM} : \frac{1}{V}$. In the same way, for the points E and G, the ratio will be $\frac{1}{V - HL} : \frac{1}{V} \propto \frac{1}{V - KN} : \frac{1}{V}$

$$:: \text{excess of elasticity of E} : \text{mean elasticity}$$

$$:: \frac{HL - KN}{V^2 - HL \times V - KN \times V + HL \times KN} : \frac{1}{V}$$

$$:: HL - KN : V.$$

Now

$$V \propto 1.$$

$\therefore$ the excess of E's elasticity $\propto$ HL — KN, and since HL — KN $= Hr : HK :: OM : OP$,

$$\therefore HL - KN \propto OM,$$

$$\therefore \text{excess of E's elasticity} \propto OM.$$

Since E and G exert themselves in opposite directions by the arc's tendency to dilate, this excess is the accelerating force of $\epsilon\gamma$, $\therefore$ accelerating force $\propto$ O M.*

ON THE HARMONIC CURVE.

Since the ordinates in the harmonic curve drawn perpendicular to the axis are in a constant ratio, the subtenses of the angle of contact will be in the same given ratio. Now the subtenses $\propto \frac{\text{arcs}}{\text{rad. of curv.}}$, and when the curve performs very small vibrations, the arcs are nearly equal.

Now the curv. $\propto \frac{1}{\text{rad.}}$, $\therefore$ subtense $\propto$ curvature.

Hence the accelerating force on any point of the string $\propto$ curvature at that point.

---

* Now bisect F f in Ω,

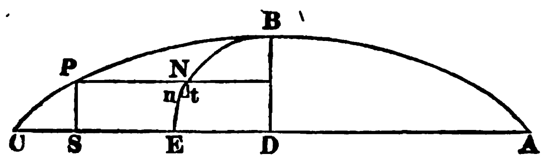
$$\therefore OM = \Omega\phi$$

For

$$OM = OP - PM = \Omega F - F\phi = \Omega\phi$$

i. e. the accelerating force $\propto$ distance from Ω the middle point. Q. e. d.

To find the equation to the harmonic curve.



Let A C be the axis of the harmonic curve C B A, D the middle point, draw B D perpendicular cutting the curve in B; draw P M perpendicular to B D cutting the curve in P, and cutting the quadrant described with the center D and radius D B in N. Draw P S perpendicular to A C.

Put

$$B D = a,\ P M = y,\ B M = x,$$

$$\therefore D M = a - x = P S.$$

$$r = \text{rad. of curv. at } B,\ B P = z,$$

$$\therefore \text{rad. of curv.} = \frac{-dz\,dx}{d^2y} \text{ (if } dz \text{ be constant).}$$

Now

$$B D : P S :: \text{curvature at } B : \text{curvature at } P$$
$$:: \text{rad. of cur. at } P : \text{rad. at } B$$

or

$$a : \overline{a - x} :: \frac{dz\,dx}{d^2y} : r,$$

$$\therefore r a d^2 y + a dz dx - x dx dz = 0,$$

$$\therefore r a dy + a dz x - \frac{x^2 dz}{2} = 0 + C.$$

Now

$$x = 0,\ dy = dx,$$

$$\therefore r a dz = 0 + C = C,$$

$$\therefore r a dy + a x dz - \frac{x^2 dz}{2} = r a dz.$$

Put

$$a x - \frac{x^2}{2} = b^2,$$

$$\therefore r a dy = \overline{ra - b^2}\, dz,$$

$$\therefore r^2a^2dy^2 = (ra - b^2)^2 \times dx^2 + r^2a^2dy^2 - 2rab^2dy^2 + b^4dy^2,$$

$\therefore (ra - b^2)^2 \times dx^2 = 2rab^2dy^2 - b^4dy^2,$

$\therefore r^2a^2dx^2 = 2rab^2.dy^2$

if (b) be small compared to (a),

$$\therefore dy^2 = \frac{radx^2}{2b^2},$$

$$\therefore dy = \frac{\sqrt{ra} \times dx}{\sqrt{2ax - x^2}} = \sqrt{\frac{r}{a}} \times \frac{adx}{\sqrt{2ax - x^2}},$$

$\therefore y = \sqrt{\frac{r}{a}} \times$ circular arc whose rad. $= a$, and vers. $= x$ $+ C$, and cor^n. $= 0$,

because when $y = 0$, $x = 0$,

$\therefore$ arc $= 0$.

$$\therefore CD = \sqrt{\frac{r}{a}} \times \text{quadrant BNE},$$

and therefore

$$\sqrt{\frac{r}{a}} = \frac{CD}{BNE},$$

$$\therefore y = \frac{BN \times \frac{1}{2}Z}{BNE}.$$

60. Prop. XLIX. Put A = attraction of a homogeneous atmosphere when the weight and density equal the weight and density of the medium through which the physical line E G is supposed to vibrate. Then every thing remaining as in Prop. XLVII. the vibration of the line E G will be performed in the same times as the vibrations in a cycloid, whose length = P S, since in each case they would move according to the same law, and through the same space. Also, if A be the length of a pendulum, since $T \propto \sqrt{L}$

The time of a vibration : time of oscillation of a pendulum A

$$:: \sqrt{PO} : \sqrt{A}.$$

Also (Prop. XLVII.), the accelerating force of E G in medium : accelerating force in cycloid

$$:: A \times HK : V \times EG;$$

since $HK : GE :: PO : V$.

$$:: PO \times A : V^2.$$

Now

$$T \propto \sqrt{\frac{1}{T}} \text{ when L is given.}$$

$\therefore$ the time of vibration : time of oscillation of the pendulum A
:: V : A
:: B C : circumference of a circle rad. = A.

Now B C = space described in the time of one vibration, therefore the circumference of the circle of radius A = space described in the time of the oscillation of a pendulum whose length = A.

Since the time of vibration : time of describing a space = circumference of the circle whose rad. = A :: B C : that circumference.

COR. 1. The velocity equals that acquired down half the altitude of A. For in the same time, with this velocity uniform, the body would describe A; and since the time down half A : time of an oscillation :: r : circumference. In the time of an oscillation the body would describe the circumference.

COR. 2. Since the comparative force or weight $\propto$ density $\times$ attraction of a homogeneous atmosphere, A $\propto \frac{\text{elastic force}}{\text{density}}$, and the velocity $\propto \sqrt{A}$.

$\propto \frac{\sqrt{\text{elastic force}}}{\sqrt{\text{density}}}$.

SCHOLIUM.

61. PROP. XLIX. Sound is produced by the pulses of air, which theory is confirmed, 1st, from the vibrations of solid bodies opposed to it. 2d. from the coincidence of theory with experiment, with respect to the velocity of sound.

The specific gravity of air : that of mercury :: 1 : 11890.

Now since the alt. $\propto \frac{1}{\text{sp. gr.}}$, $\therefore$ 1 : 11890 :: 30 inches : 29725 feet = altitude of the homogeneous atmosphere. Hence a pendulum whose length = 29725, will perform an oscillation in 190″, in which time by Prop. XLIX, sound will move over 186768 feet, therefore in 1″ sound will describe 979 feet. This computation does not take into consideration the solidity of the particles of air, through which sound is propagated instantly. Now suppose the particles of air to have the same density as the particles of water, then the diameter of each particle : dis-

tance between their centers : : 1 : 9, or 1 : 10 nearly. (For if there are two cubes of air and water equal to each other, D the diameter of the particles, S the interval between them, S + D = the side of the cube, and if N = N°. N S + N D = N°. in the side of the cube, N°. in the cube $\propto N^3$. Also, if M be the N°. in the cube of water, M D the side of the cube and the N°. in the cube $\propto M^3$.

$$\text{Put } 1 : A :: N^3 : M^3,$$

$$\therefore M = A^{\frac{1}{3}} N,$$

By Proposition

$$N S + N D = M D = N A^{\frac{1}{3}} D,$$

$$\therefore S = D \times \overline{A^{\frac{1}{3}} - 1},$$

$$\therefore S : D :: A^{\frac{1}{3}} - 1 : 1,$$

$$\therefore S + D : D :: A^{\frac{1}{3}} : 1 :: 9 : 1 \text{ if } A = 870$$

$$\text{or } 10 : 1 \text{ if } A = 1000).$$

Now the space described by sound : space which the air occupies : : 9 : 11, $\therefore$ space to be added $= \frac{979}{9} = 108$ or the velocity of sound is 1088 feet per 1″.

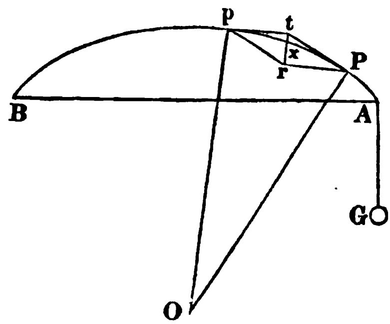
Again, also the elasticity of air is increased by vapours. Hence since the velocity $\propto \frac{\sqrt{\text{elasticity}}}{\sqrt{\text{density}}}$; if the density remain the same the velocity $\propto \sqrt{\text{elasticity}}$. Hence if the air be supposed to consist of 11 feet, 10 of air, and 1 of vapour, the elasticity will be increased in the ratio of 11 : 10, therefore the velocity will be increased in the ratio of $11\frac{1}{2} : 10\frac{1}{2}$ or 21 : 20, therefore the velocity of sound will altogether be 1142 feet per 1″, which is the same as found by experiment.

In summer the air being more elastic than in winter, sound will be propagated with a greater velocity than in winter. The above calculation relates to the mean elasticity of the air which is in spring and autumn. Hence may be found the intervals of pulses of the air.

By experiment, a tube whose length is five Paris feet, was observed to give the same sound as a chord which vibrated 100 times in 1″, and in the same time sound moves through 1070 feet, therefore the interval of the pulses of air = 10.7 or about twice the length of the pipe.

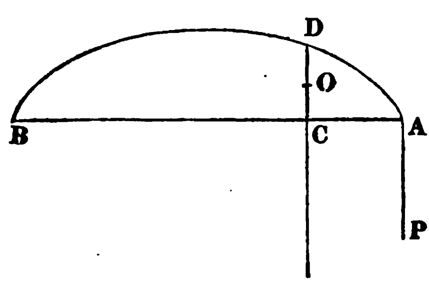
62. On the vibrations of a harmonic string.

The force with which a string tends to the center of the curve : force which stretches the string : : length : radius of curvature. Let P p be a



small portion of the string, O the center of the curve; join O P, O p, and draw P t, p t, tangents at P and p meeting in t, complete the parallelogram P t p r. Join t r, then P t, p t represent the stretching force of the string, which may be resolved into P x, t x and p x, t x of which P x, p x destroy each other, and 2 t x = force with which the string tends to the center O. Now the $\angle$ t P r = $\frac{1}{2}$ $\angle$ P O p, $\therefore$ $\angle$ t P x = $\angle$ P O p, $\therefore$ t r : P t : : P p : O P, i. e. the force with which any particle moves towards the center of the curve : force which stretches it : : length : radius.

63. To find the times of vibration of a harmonic string.



Let w = weight of the string. L = length.

D d : L : : weight D d : w

$$\therefore \text{weight of D d} = \frac{\text{D d} \times \text{w}}{\text{L}}$$

Also

$$D\,d : \frac{L^2}{p^2 a^2} = \text{rad. of curve} :: \text{the moving force of } D\,d : P$$

$$\therefore \text{the moving force of } D\,d = \frac{P \times D\,d \times a\,p^2}{L\,w}$$

$$\therefore \text{accelerating force} = \frac{P \times D\,d \times a\,p^2}{L^2} \times \frac{L}{D\,d \times w}$$

$$= \frac{P \times a\,p^2}{L\,w.}$$

if $D\,O = x$, $D\,C = a$, $O\,C = a - x$,

$$\therefore \text{the accelerating force at } O = \frac{P\,p^2 . \overline{a - x}}{L\,w}$$

$$\therefore v\,d\,s = \frac{g . P\,p}{L\,w} \times \overline{a\,d\,x - x\,d\,x}$$

$$\therefore v^2 = \frac{g\,P\,p^2}{L\,w} \times \overline{2\,a\,x - z^2}$$

$$\therefore v = \sqrt{\frac{g\,P\,p^2}{L\,w}} \times \sqrt{2\,a\,x - x^2}.$$

$\therefore$ C and $1 = 0$,

$$\therefore d\,t = \frac{d\,x}{v} = \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \frac{d\,x}{\sqrt{2\,a\,x - x}}$$

$$\therefore t = \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \text{cir. arc rad.} = 1$$

and

$$\text{vers. sine} = \frac{x}{a},$$

when $x = a$,

$t = 0$.

$$\therefore \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \text{quadrant} = \sqrt{\frac{L\,w}{g\,P\,p^2}} \times \frac{P}{2}$$

$$= \tfrac{1}{2} \times \sqrt{\frac{L\,w}{g\,P^2}}$$

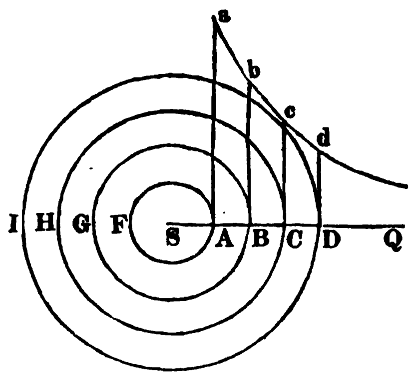
$$\therefore \text{time of a vibration} = \sqrt{\frac{L\,w}{g\,P}}\ 1''$$

$$\therefore \text{number of vibrations in } 1'' = \sqrt{\frac{g\,P}{L\,w}}.$$

Cor. Time of vibration = time of the oscillation of a pendulum whose length $= \dfrac{L\,w}{P\,p^2}$.

For this time $= \sqrt{\frac{L w}{g P}}$.

64. PROP. LI. Let A F be a cylinder moving in a fluid round a fixed axis in S, and suppose the fluid divided into a great number of solid



orbs of the same thickness. Then the disturbing force $\propto$ translation of parts $\times$ surfaces. Now the disturbing forces are constant. $\therefore$ Translation of parts, from the defect of lubricity $\propto \frac{1}{\text{distance}}$. Now the difference of the angular motions $\propto \frac{\text{translation}}{\text{distance}} \propto \frac{1}{\text{distance}^2}$. On A Q draw A a, B b, C c, &c. : : $\frac{1}{\text{distance}^2}$, then the sum of the differences will $\propto$ hyperbolic area.

$\therefore$ periodic time $\propto \frac{1}{\text{angular motion}} \propto \frac{1}{\text{hyperbolic area}} \propto$ distance.

In the same way, if they were globes or spheres, the periodic time would vary as the distance [2].

END OF THE FIRST VOLUME.

[Sect. I]

round

per of soli

ation of

ransla-

differ-

draw

will

ime